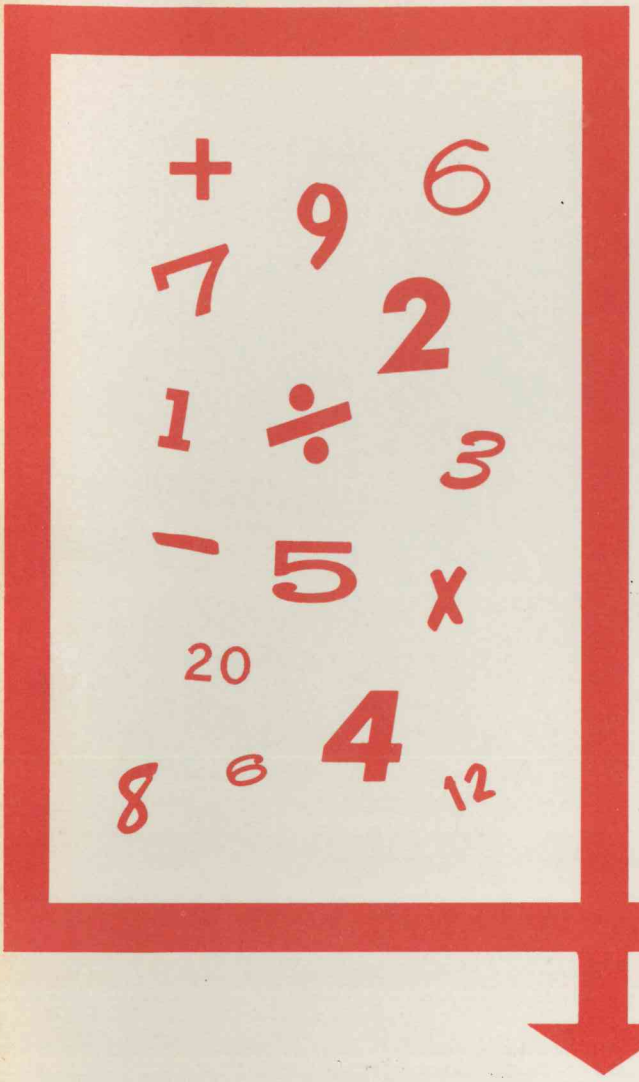

ICS®

Practical Arithmetic

Square Root and Number Systems

By

W. M. SPANGLER, M.A.

6750D-1

Part 4

Edition 1

Practical Arithmetic

PART 4

Speeding Up and Checking Calculations Powers, Roots, and Number Systems

By

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Serial 6750D-1

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What This Text Covers . . .

This text advances your knowledge and understanding of the basic operations of addition, subtraction, multiplication, and division. In it you will find that we put emphasis upon the why *of what you are doing. Make certain that you understand these explanations, because they are very important. Many new ideas, words, and topics will be covered here, so study each section carefully.*

1. SPEEDING UP AND CHECKING THE BASIC OPERATIONS ..... Pages 1 to 30

   You will learn methods that can be used to speed up your computations. Checking your work is important. In this section we tell you how to check it. Casting out nines and casting out elevens are given as general methods of checking. The meanings of the commutative, associative, and distributive properties are included. Symbols of grouping and order of operations are explained also.

2. A BETTER UNDERSTANDING OF FRACTIONS AND DECIMALS ..... Pages 31 to 40

   Several important principles of arithmetic are used to explain the methods of working with fractions. The meaning of equivalent fractions is emphasized. Also included is another look at decimals and a discussion of per cent.

3. POWERS AND ROOTS OF NUMBERS ..... Pages 41 to 62

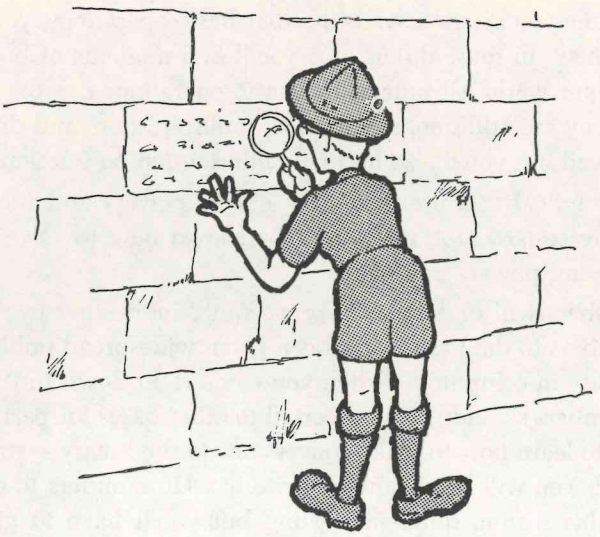
   This section shows how to find powers of numbers, especially squares and cubes of numbers. The meanings of "base number" and "exponent" are also explained, and writing exponents correctly is emphasized. The last part of this section deals with the method of calculating the square roots of whole numbers, fractions, and decimals.

4. APPLICATIONS OF SQUARES AND SQUARE ROOTS ..... Pages 63 to 65

   Here you will see how we use squares and square roots in working with some common figures: the circle, the square, and the right triangle.

5. NUMBER SYSTEMS ..... Pages 66 to 76

   A different explanation of our number system is given here. The similarities of other number systems to the decimal system are explained. You will learn how to change numbers from one system to another system. An important modern topic, the binary system, is also discussed. This is the language of computers and is most vital because of the role that computers play in our lives.



Practical Arithmetic

PART 4

Speeding Up and Checking Your Calculations

A Little Knowledge Is Dangerous

1. Now that you have opened this text you have reached a very important milestone in your search for more knowledge. At this point either you have completed the first three *Practical Arithmetic* texts or you have passed a test and are permitted to begin your study of practical arithmetic with this text. Nowhere in your study will it be more important for you to realize the truth expressed in, "A little knowledge is a dangerous thing." Don't think that you know all there is to know simply because you have moved beyond the first three texts of the series. Neither should you think that you will be getting the "same old thing." Your attitude toward arithmetic must be a receptive one; you must know full well that arithmetic encompasses much more than merely being able to add, subtract, multiply, and divide. You should adopt as your motto, "Let knowledge grow from more to more." If this is your attitude, you will be eager to turn the pages of this text to learn more about arithmetic and its uses.

As you begin your study of *Practical Arithmetic*, Part 4, remember that there is much more to learn about the four basic operations. Although there is a review at the beginning of this assignment, you should not allow this review to fool you into thinking that there's nothing different coming. Perhaps the best way to convince you that new topics are to be discussed in this text is to tell you what you are going to study.

Topics to Be Studied

2. Even while you are reviewing the four basic operations, you will be learning something new. In these discussions you'll find methods of both speeding up and checking your work. All four of the basic operations can be, and should be, checked. Not only are addition, subtraction, multiplication, and division of whole numbers reviewed but you'll also find new information on fractions and decimals.

One very practical and useful section covers powers and roots. You'll learn how to calculate square root, and you'll be shown how to solve some practical problems involving powers and roots.

Binary numbers will undoubtedly prove quite interesting to you. The binary system, or numbers to the base 2, has been given widespread publicity these days because of its use in computers. When you come right down to it, there isn't any reason why numbers shouldn't be converted to other bases for particular purposes. You are going to learn how to make conversions to the binary system and to other systems as well. You will learn how to write base-10 numbers to other bases and you will find this section quite interesting but you'll have to give it your full attention.

Special Features to Look For

3. There are some important features in this text. They have been stressed throughout, and you too should focus special attention upon them. First of all, notice particularly the reference made to "modern mathematics." Whenever you meet this phrase, you should stop and try to see exactly what it is that makes this math "modern." You'll soon find out that it's the "same old math." The so-called new approach merely explains some of the things that were being done even before your grandfather's time. Your grandfather probably didn't really know *why* some of the rules worked or *how* they worked. But now you will have the benefit of learning the mathematical principles upon which these rules are based.

For another thing, there is emphasis upon estimating the correct answer. There are signposts throughout the text pointing in the right direction, but it's up to you to follow them. Common sense should tell you when you have arrived at an outlandish or ridiculous answer. You should be able to judge when your answer is not reasonable. This is really not as difficult as it may seem. When a pianist strikes a wrong note, he knows immediately that he has done something wrong. So it should be with you! When you arrive at a wrong answer, your "number sense" should tell you that it is wrong. Be watchful and alert to pick up any errors in your work.

Finally, your study of this text has been augmented by the sections entitled Check Your Learning. If you have completed the first three texts, you are already familiar with the advantages of this kind of instruction. Those of you who are beginning at this point have a new experience in learning awaiting you. The ICS now offers this latest concept in teaching.

Now you are ready to go ahead with this text. The few topics on which you were briefed and the special features to look for should make you want to continue immediately to all of the new things you will learn about arithmetic. If you have taken heed to the motto, "Let knowledge grow from more to more," you should have your eyes focused on the next article.

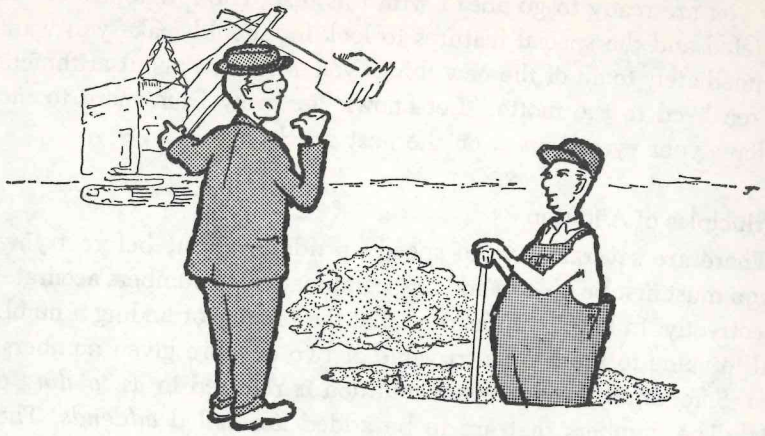
Basic Principles of Addition

4. There are several ways to speed up addition, but before trying to gain speed you must first be able to add any combination of numbers accurately, which means correctly. In Part 1 you learned that the process of finding a number which is equal in value to the combined value of two or more given numbers is called *addition.* Sometimes the process of addition is referred to as *totaling* or *finding the total.* The numbers that are to be added are called *addends.* The number obtained by performing addition is called the *sum* or *total.*

You learned earlier that in adding two numbers the order in which you added them was not important. For example, $2 + 4 = 6$ and $4 + 2 = 6$. This fact, though it has always been used in arithmetic, is now receiving considerable attention in what is called modern mathematics. The fact that the order in which you add two numbers does not affect their sum is known as the *commutative property of addition* and is more commonly called the *law of order.* Likewise, when you were adding several numbers in the past you must have recognized that you could group them together in any way without changing the sum. For example, to find $4 + 3 + 7$, it doesn't matter whether you add the 3 to the 4 first and then add that sum to the 7 or if you add the 7 to the 3 first and then add that sum to the 4. Stating these observations in the form of equations, they become $(4 + 3) + 7 = 14$ and $4 + (3 + 7) = 14$. Here the parentheses are used to indicate which numbers are to be added first. This fact too — that it doesn't matter in what order you add — has been given considerable attention in modern mathematics. It is called the *associative property of addition,* but it is more commonly known as *the law of grouping.*

In the teaching of modern mathematics, stress is put upon the fact that addition is an operation that is both commutative and associative. It is these properties, as well as other important properties (many of which will be mentioned later in this text), that "modern mathematics" emphasizes. As pointed out before, these ideas have always been important and many of them have always been used in the study of arithmetic. You were using these properties long before you knew what they were called. Knowing them will help you understand the "whys" of arithmetic. Used together, the commutative and associative laws for addition tell us that *we can regroup or rearrange addends any way we like without changing the sum of the numbers.* For example,

$$(14 + 9) + (6 + 1) = (14 + 6) + (9 + 1) = (9 + 1) + (6 + 14) = 30$$



Speeding Up Addition

5. We will show you how the commutative and the associative properties are used to speed up addition. It is always a good idea to combine digits that add up to 10 in finding a sum, because 10's are added easily either to themselves or to other numbers. For example, to add the numbers 7, 19, and 3, it is easier to add 7 to 3 to get a sum of 10 and then to add the 10 to the 19, for a total of 29, than it would be to add the 7 to the 19 and then add that sum to the 3. Also, in adding the numbers 14, 3, 6, and 7, if you recognize that these numbers can be written as $(14 + 6) + (3 + 7)$, the addition is much simpler than adding the numbers in the order in which they appear.

Combining numbers that add to 10 can be applied to simplify addition in column form. For example, to find the sum of the numbers 149, 72, 31, 188, and 13 we write the numbers in a column like this:

```
149
 72
 31
188
 13
453
```

Notice that in the units column we have the digits 9 and 1 and 8 and 2. By adding these pairs together and then to 3, the sum of the units column is 20 + 3 or 23. The 3 is written in the units place in the total, and the 2 is carried to the tens column. In the tens column take the 2 carried from the units column and the 8 for one combination and the 7 and 3 for another combination. Adding these pairs and then combining the sum with the digits 1 and 4, you have 25. The 5, of course, is written in the tens place of the total, and 2 is carried to the hundreds column. The sum of the hundreds column is then written down in the total. *Plainly, then, to speed up addition, group numbers which can be added together quickly.* Practice will definitely help you gain speed in adding.

Subtotal Method

6. It is also possible to speed up the addition of long columns of figures by separating the numbers into groups, drawing a line under four or five numbers,

and totaling each group separately. Totals such as these are called *subtotals.* Then we find the sum of the subtotals for the answer.

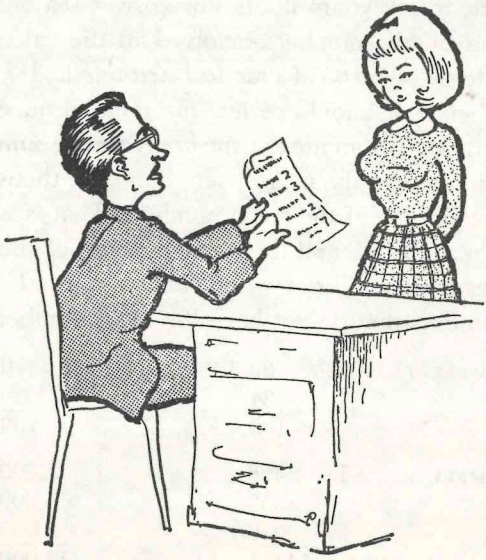
1814		
736		
983		
621	4154	subtotal
801		
373		
98		
67		
83	1422	subtotal
5576	Ans.	

In the example we have drawn a line under the first four numbers and under the next five numbers. The subtotal of the first four numbers (4154) is written above the first line, and the subtotal of the last five numbers (1422) is written above the bottom line. The answer is the sum of these two subtotals.

This method for speeding up addition is recommended primarily for finding the total of a long column of numbers. There is no particular advantage in using the subtotal method for adding just a few simple numbers. You yourself can devise other methods of your own to speed up addition, and you can certainly find many ways for practicing addition. As you are going about your duties or commuting from place to place, you can use familiar objects to practice finding a sum. For example, add the digits on a license plate, then add this total to the total of another license plate, and so on. Add several street addresses and total several purchases made in a store. These are some of the ways in which you can practice to improve your speed.

Checking Addition

7. You should always go over your work to make sure that you have not made a mistake. Going over your own work to make sure it is correct is called *checking* your work. In a real-life situation when you are required to do addition, the



answer will seldom, if ever, be given, but the correct result is always essential. You should, of course, be careful while you are adding. But to make sure the sum is correct, you should check your work. We will now see how addition can be checked.

Earlier in this text you learned that the order in which numbers are added is not important. This fact should suggest our first method of checking. If you found the sum by adding the digits in each column from the top to the bottom, check your result by adding the digits in each column from the bottom to the top. In other words, for your check, add in the direction opposite to the direction in which you added the first time. Checking the work in this manner is much better than just repeating what you have already done. The main point is that if you add in the same direction both times, you may make the same mistake both times!

When you add a column of digits in both directions (up and down) and get the same sum both times, your result is probably correct. If you get two different answers, you should add the digits again in both directions. Addition is very simple, but if you try to add too fast, you are likely to make mistakes. Getting the correct answer is always more important than speed! Your speed can be improved with practice.

Another good way to check addition is to use the method of adding by subtotals. It is especially helpful when a long column of numbers has been added. However, if you added by this method the first time, then for a check you should use different subtotals. The subtotal method is explained in Art. 6.

If you work carefully and check your work, in most instances you can be sure that your answer is correct. However, there is another kind of checking that you should consider. It is always a good policy after performing any operation of arithmetic to ask yourself, "Is this answer reasonable?" To answer this question, we round off the numbers involved in the calculations. This is something you learned to do in Part 3 of *Practical Arithmetic*. If you have forgotten how to round off numbers, you should review the topic at once.

To quickly get an idea of the *approximate sum* of numbers, we round off each of the numbers to the nearest ten, hundred, thousand, or ten thousand according to the magnitude of the given numbers. That is, we round off each number at its highest place value and then find the sum of the rounded-off numbers. The following examples will show you what we mean. Notice that in Example 5 we have used numbers that do not have the same number of digits.

EXAMPLE 1.	Add	69		70	nearest ten
		34		30	
		103	Ans.	100	approximate sum
EXAMPLE 2.	Add	2468		2,000	nearest thousand
		3957		4,000	
		8046		8,000	
		14471	Ans.	14,000	approximate sum

EXAMPLE 3.

Add	43785		40,000	nearest ten thousand
	86058		90,000	
	129843	Ans.	130,000	approximate sum

EXAMPLE 4. Mr. Albert earns $4700 per year, Mr. Aiken earns $5640 per year, and Mr. Samuels earns $8100 per year. What is the actual and what is the approximate annual income of these three men?

$ 4700			$ 5,000	nearest thousand
5640			6,000	
8100			8,000	
$18440	actual income	Ans.	$19,000	approximate income

EXAMPLE 5.

Add	3235		3000	
	786		800	
	49		50	
	4070	Ans.	3850	approximate sum

Note that 3850 rounds off to 4000, which is close enough to 4070 to provide a check of the *reasonableness* of our answer.

There is one very important point about this method of checking that you probably have noticed for yourself, but we want to emphasize it here. Do not be concerned if your approximation is somewhat smaller than or larger than the actual value. The important point is that this method is not intended to be an accurate check; it is only an answer to the question, "Is the result reasonable?" For example, if in adding the numbers, 742, 519, 896, and 221 you arrived at 3378 for the sum, you would immediately know that the addition was wrong, since the check approximation 700 + 500 + 900 + 200 is 2300. This method of checking is both convenient and fast.

Check Your Learning

One of the newest concepts in educational methods has been adapted for use in the "Practical Arithmetic" series. This new approach to learning is known as programmed instruction. It represents a milestone in both teaching and learning methods. You've probably already heard a good deal about this new concept. Here is your chance to put it to work for yourself. We at ICS are proud to make it available to you.

Used properly, the Check Your Learning we've included in these texts will speed your learning process and impress the most important points of information more firmly on your mind. By means of these statements, you are exposed a second time to the essential facts and the important procedures explained in the text material. All this is accomplished without the painful process of "hunting and finding" to pinpoint the most important facts. You'll also find that the Check Your Learning provides an easy means of self-testing on important material.

There are groups of statements, numbered consecutively, scattered throughout this text. Most of these statements require you to supply a one- or two-word answer. Others give you a choice of two words, and you must select the word you think is correct. You'll find the correct answer slightly below and to the right of each statement. Do not look at our answer until you've tested your understanding of a statement by completing it yourself. Cover the answer with a piece of paper and write your own answer in the blank provided in the statement. Then compare your answer with the one given at the right. If your answer is correct, go on to the next statement. If your answer is wrong, review the text material on which the statement is based until you understand your error.

1. Combining two or more numbers to find the sum is called _____.

(addition)

2. The numbers which you add are called _____.

(addends)

3. Since $5 + 3 = 8$ and $3 + 5 = 8$ it doesn't matter in which order you add the numbers because the sum isn't affected. This fact is called the Commutative Property of Addition or the _____ _____ _____.

(Law of Order)

4. Another name for the Associative Property of Addition is the _____ _____ _____.

(Law of Grouping)

5. When addends are re-arranged it (does, does not) affect their sum.

(does not)

6. _____ _____ emphasizes that addition is an operation that has both commutative and associative properties.

(Modern mathematics)

7. In adding a long column of numbers if you combine numbers that add up to _____ it will simplify your addition.

(ten or 10)

8. To speed up addition _____ _____ which can be added together quickly.

(group numbers)

9. To speed up the addition of long columns of figures, separate the numbers into groups and then total each group separately. The sum of each of these groups is called a _____ _____.

(subtotal)

10. When you go over your work for the second time using the method you previously used, it is called _____ your work.

(checking)

11. You (can, cannot) check your work on addition.

(can)

12. If you added a column of digits in both directions, that is, up and down, and you obtained the same sum each time, your answer is probably (correct, incorrect).

(correct)

13. When you round off each number at its highest place value and then find the sum of these numbers your answer will be the _____ sum.

(approximate)

14. When checking addition by approximation, the approximate answer is often smaller than or larger than the actual sum of the numbers. This method (is, is not) intended to be an accurate check but is used to find out whether your answer is reasonable.

(is not)

Subtraction

8. Subtraction is a comparatively simple operation because it involves only three numbers. These numbers are the number from which you are to subtract, the number to be subtracted, and the difference. However, most persons make more mistakes in subtraction than in addition. Perhaps this is true because they don't get enough practice subtracting.

Subtraction is the process of finding how much is left after one number has been taken away from another number. The number from which another number is to be taken is called the *minuend.* In arithmetic this number is always the larger of the two numbers. The number which is taken away from the larger one is called the *subtrahend.* The result obtained by subtracting is called the *difference.* The sign used to indicate subtraction is −, which is read *minus.*

Subtraction is the inverse or opposite of addition and differs in many respects from addition. Subtraction, unlike addition, does not possess the *commutative* property, because 5 − 3 is not the same as 3 − 5. In other words, the numbers cannot be interchanged in subtraction. Furthermore, subtraction does not possess the *associative property* as addition does. If you think about this fact for awhile, you will probably realize that it is true. For example, 15 − 6 − 3 has a different value depending upon whether you subtract 3 from the difference of 15 − 6 or if you subtract the difference of 6 − 3 from 15. By doing it the first way you get 6, and by doing it the second way you get 12. Of course you know that 6 is the correct answer. This, like many of the other properties of modern mathematics, is something that you have known about and used before.

If you can add easily, it will be a great help with subtraction. The best way to increase your speed in subtraction is to practice it until finding the difference between two numbers becomes an automatic process for you. Practice and practice some more! Try to make up problems involving subtraction from your everyday situations just as you did for practice in addition.

Remember, as previously explained in Part 1, Arts. 31 to 34, that there are two different methods of subtracting. The first way that we taught was to take the subtrahend away from the minuend, and the second one was to find the number you must add to the smaller number to obtain the larger number. This latter method is sometimes called the the additive method of subtraction.

Subtraction Without Borrowing

9. Many people find borrowing in subtraction quite troublesome. If you are one who dislikes borrowing, perhaps you'll want to try the following method which eliminates borrowing. Suppose you are to subtract 39 from 84. When 9 is subtracted from 14, the result is 5; so you write 5 in the units place of the difference. Now add 1 to the 3 in the tens place of the subtrahend and consider this number as 4. Your next subtraction will be 4 from 8 or 4. Thus the difference or answer is 45. This subtraction was performed without borrowing in the minuend.

Subtraction without borrowing can be done in another way, called *add and carry*. The method is so called because you add and carry just as you do in addition problems. Let's begin this explanation by reminding you of the additive method of subtraction. You ask yourself, "What number must be added to the smaller number to get the larger number?" With this idea in mind we'll again subtract 39 from 84. Write the subtrahend below the minuend as in other subtraction problems. You know that 9 and 5 are 14, so you write the 5 in the units column of the difference and carry the 1 to the tens column of the subtrahend. Now, the tens figure in the subtrahend becomes 3 + 1 or 4, and so you ask what number added to 4 gives 8. The answer is 4, and this number is written in the tens column of the difference. Therefore, 84 − 39 = 45.

Take another example. Let's subtract 42 from 237. Here is the way you would do the work.

$$\begin{array}{r} 237 \\ \underline{42} \\ 195 \end{array} \quad \text{Ans.}$$

What number added to 2 gives 7? The answer is 5, and this number is placed in the units column of the difference. What number added to 4 gives 13? This result is 9, which is placed in the tens column of the difference. Since 9 + 4 = 13, you carry 1 to the hundreds column of the subtrahend, which now becomes 0 + 1, or 1. Finally, what number added to 1 gives 2? The answer, of course, is 1, and the difference in this problem is 195.

When subtracting with the add-and-carry method, don't make the mistake of carrying a 1 to any place of the subtrahend when it isn't necessary to "borrow." For instance, if you were subtracting 32 from 237 you would not add anywhere in the subtrahend because it would not be necessary to borrow.

Checking Subtraction

10. The fact that subtraction is the reverse of addition makes it very easy to tell whether your answer is correct. To check subtraction, you add the difference, or your answer, to the subtrahend. The sum should be the same as the minuend.

This method is the most desirable one for checking subtraction, and you should make a habit of using it so that checking will become automatic. Here are a few illustrations of the procedure.

EXAMPLE 1.

minuend	7321	
subtrahend	4567	
difference	2754	Ans.
CHECK	7321	

EXAMPLE 2.

$$\begin{array}{r} 39785 \\ -24688 \\ \hline 15097 \\ \text{CHECK} \quad 39785 \end{array} \quad \text{Ans.}$$

Finally, you should check a difference by answering the question "Is my answer reasonable?" Now we are trying to do for subtraction exactly what we did with approximations in addition. With the question "Is my answer reason-

able?" we want you to develop a "number sense." In other words, by the use of approximations you should be able to recognize the correctness of your answer. Develop such a fine number sense that you can almost instinctively tell if your results are sensible.

EXAMPLE 3.	Subtract	7321		7000	
		4588		5000	
		2733	Ans.	2000	approximate answer
EXAMPLE 4.	Subtract	46534		50,000	
		17840		20,000	
		28694	Ans.	30,000	approximate answer

Check Your Learning

15. How many numbers do problems in subtraction involve?

(three)

16. Subtraction is the process of finding how much is left after one number has been _____ _____ from another number.

(taken away)

17. In subtraction the larger number is called the _____.

(minuend)

18. Which sign is used to indicate subtraction?

(−)

19. Subtraction is the (same as, opposite) of addition.

(opposite)

20. The associative property (does, does not) apply to subtraction.

(does not)

21. In solving subtraction problems by the use of the add and carry method you (do, do not) borrow from the minuend.

(do not)

22. The best check for subtraction is to add the _____ to the _____. This sum should equal the _____.

(difference; subtrahend; minuend)

23. When checking by the approximate method the difference when 99 is subtracted from 445 is _____.

(300)

Practice Problems

Write your answers to the following problems on scratch paper. Compare your answers with the answers given at the back of this instruction text. Sometime later, write the answers to all the problems once more as a review. These answers are *not* to be submitted to the Schools.

1. Add the following groups of numbers. Check your answers by adding in the direction opposite to the one you used for obtaining the sums the first time.

a)	*b)*	*c)*	*d)*	*e)*
373	67	321	7659	76439
69	324	9	31	9907
582	1301	512	955	62015

2. Add the following columns. Separate the numbers into groups of three, find the subtotals, then add these subtotals for the final result.

a)	*b)*	*c)*	*d)*	*e)*
171	6634	3621	288	526
388	5622	4784	397	642
963	3635	320	965	7489
565	8844	5639	1676	1322
666	7600	8888	2299	28
729	9843	791	8865	64
111	2772	6419	9709	421
963	6855	567	24	10007
579	1277	21	6666	4055

3. Perform the following additions. Check your work by approximating the numbers.

a)	*b)*	*c)*	*d)*	*e)*
17	313	5167	5559	222
23	265	952	661	4763
	389	9345	4267	62
				876

4. Subtract in the following problems. Check each answer by adding the difference to the subtrahend.

a)	*b)*	*c)*	*d)*	*e)*
898	677	2996	17030	9475
323	129	1497	16065	3766

5. Perform the following subtractions. Give the approximate answer to these problems also.

a)	*b)*	*c)*	*d)*	*e)*
79567	63576	67059	59678	787032
33677	37005	30583	21314	197039

6. What was the total attendance at the recent Brooklyn–New York 7-game World Series if 63,869 persons saw the first game, 64,707 the second, 34,209 the third, 36,242 the fourth, 36,796 the fifth, 64,022 the sixth, and 62,465 the seventh game?

7. In 1964, the circulation of our city library for January was 176,325 books, for February 93,000 books, for March 99,765 books, and for April 111,076 books. How many books came in and out of this library for the first four months of this year? Be sure to give the approximate answer also.

8. Last year, Mr. Jones drove his car 19,971 miles. Mr. Symons traveled 13,356 miles throughout the year. How many more miles did Mr. Jones travel? Don't forget to give the approximate answer to this problem also.

Multiplication

11. As you know, multiplication is a quick way of adding several equal

numbers. The number that is to be multiplied is called the *multiplicand*; the number by which you multiply is called the *multiplier*. The result of multiplication, no matter how many numbers are involved, is called the *product*. The numbers that are multiplied together are called the *factors* of their product. The symbol $\times$ indicates the operation of multiplication.

Speeding Up Multiplication

12. Besides skill obtained by constant practice, there are a few shortcut methods for speeding up multiplication. One of the most important of these is the shortcut for multiplying by 10. To multiply a number by 10, you just put a zero on the end of the multiplicand to get the product. Thus, $10 \times 358 = 3580$ and $10 \times 4750 = 47500$. Since you should think of every number, even the integers, as having a decimal point, it is better to think of the shortcut for multiplying by 10 as follows:

RULE. *To multiply a number by 10, move its decimal point one place to the right.*

The following examples will illustrate this rule.

$10 \times 52. = 520.$
$10 \times 37.4 = 374.$
$10 \times 0.3625 = 3.625$
$10 \times 0.041 = 0.41$
$10 \times 470. = 4700.$
$10 \times 32.65 = 326.5$

Multiplying by 100, 1000, and So Forth

13. The shortcut method for multiplying by 10 can be extended to provide a shortcut for multiplying when the multiplier is 100, 1000, or 1 followed by any number of zeros. To multiply by 1 followed by several zeros, as, for example, by 100 or 1000, just move the decimal point in the multiplicand as *many places* to the right as there are zeros in the multiplier. This will give you the product. Thus

$$100 \times 375.4 = 37540.$$
$$1000 \times 0.126 = 126.$$
$$10000 \times 2.37 = 23700.$$

Multiplying by Any Number Ending in Zero

14. Basically, the shortcut method of the preceding article can be applied whenever the multiplier is a number ending in zero or zeros. In other words, we can use the method of multiplying by 100 or 1000 when the multiplier is 20, 30, 90, 400, or 7000. This is true because numbers such as 20, 30, 90, 400, and 7000 can be considered as 2×10, 3×10, 9×10, 4×100, and 7×1000. Then it is a simple matter to think of multiplying by 20 as multiplying by 2 and then by 10. Even though we perform two operations (multiplying by 2 and then by 10) instead of a single operation (multiplying by 20) there is a saving. This is true

because the two operations are such simple ones. Actually, in practice you will soon learn to do both so fast that you will not think of them as two distinct operations.

Let us suppose that we want to multiply 200×214. We first multiply 214 by 2 and get 428. Then we add the two zeros and get 42800. This is the answer. Next let's do 600×437. Here $6 \times 437 = 2622$ and $2622 \times 100 = 262200$. Isn't that simple? Try the following for yourself for practice.

Practice Problems

1. 50×37
2. 400×5.12
3. 700×397
4. 47.53×900
5. 839×600
6. 3.1×20

Commutative and Associative Properties of Multiplication

15. In Part 1 of *Practical Arithmetic* you learned that it makes no difference which of two numbers is considered to be the multiplier. The product is the same in either case. For example, $7 \times 8 = 56$ and $8 \times 7 = 56$. You will recognize that this fact is very similar to the commutative property that we met in addition. Actually, multiplication, as you might suspect, possesses the commutative property. This means that all the factors of a product can be arranged in any order you want.

At this point, since the commutative property applies, we're sure you will wonder about the associative property. Does multiplication possess the associative property? Let's try it. Take $2 \times 3 \times 8 \times 7$. You will get the same answer if you say 6×56, 14×24, 16×21, or 48×7. While this is not a proof that the associative law holds for multiplication, it does give a reason for thinking that it does. And in modern mathematics it *is* proved that the associative law, as well as the commutative law, applies to multiplication.

These two properties, the commutative and associative properties, can be used to simplify multiplication in much the same way that we used them to simplify addition. Suppose, for example, we want to find $4 \times 3 \times 19 \times 25$. By rearranging these factors and applying the rule for multiplying by 100, we can find the product quickly and easily. We can multiply in the following way:

$$\begin{aligned} &4 \times 3 \times 19 \times 25 \\ &(4 \times 25) \times 19 \times 3 \\ &100 \times 19 \times 3 \\ &1900 \times 3 = 5700 \end{aligned}$$

Other Shortcuts in Multiplication

16. There are other shortcut methods of multiplying numbers. Here is a list of some that will be of most benefit to you. It will be well worthwhile to carefully study the following shortcuts and practice them whenever possible.

a) To multiply a number by 50, multiply it by 100 and divide that product by 2.
b) To multiply a number by 25, multiply it by 100 and divide that product by 4.
c) To multiply a number by 20, multiply it by 100 and divide that product by 5.
d) To multiply a number by $33\frac{1}{3}$, multiply it by 100 and divide that product by 3.
e) To multiply a number by $12\frac{1}{2}$, multiply it by 100 and divide that product by 8.
f) To multiply a number by $16\frac{2}{3}$, multiply it by 100 and divide that product by 6.

You will notice that in *(a)* and *(c)* you have another method of multiplying by a number ending in zero. See Art. 14, and use whichever method you prefer.

Ways to Check Multiplication

17. Checking your work in multiplication is just as important as checking your work in addition and subtraction. Most of your errors can be detected if you make it a practice to check your work. Frequently, however, the same error is repeated in checking multiplication. Therefore, in checking multiplication use methods different from the ones that you used to obtain the answer. You will recall that we discussed this same point in connection with checking addition.

Using the multiplicand as the multiplier and the multiplier as the multiplicand is the most common way of checking a multiplication problem. Because the commutative and associative properties apply to multiplication, we know that interchanging multiplicand and multiplier will not affect the product. Thus,

EXAMPLE 1.

719			238	
238			719	
5752			2142	
2157			238	
1438			1666	
171122	Ans.	CHECK	171122	Ans.

Transposing the multiplicand and the multiplier is a good way to check a multiplication problem. It is both easy to apply and easy to remember. Furthermore, for the beginner it has the added advantage of offering more practice in multiplication.

Finally, don't forget about checking multiplication by estimating the result just as we did when checking addition and subtraction. Basically the procedures are very similar. Round off the multiplicand and the multiplier at their highest place values and find the product of these rounded off numbers. Here is an example.

EXAMPLE 2.

974		1,000	
264		300	
3896		300,000	approximation
5844			
1948			
257136	Ans.		

As you have been proceeding, you should have been constantly sharpening

and increasing your number sense. Now that you have reached this stage of your training it should not be necessary for us to ask you, "Is your answer reasonable?" This question should have already been in your mind. And you should have answered it. For instance, if in the above example you had obtained an answer of 215236, you should have done the problem again, because 215000 is too far from 300,000.

Check Your Learning

24. The product of 6 and 100 is formed by writing 6 followed by _____ zero(s).
(two)

25. When you multiply a number by 10, 100, 1000, and so forth, first move the decimal point as many places to the (left, right) as there are zeros in the given multiplier.
(right)

26. When you are multiplying, order (does; does not) matter.
(does not)

27. In modern mathematics it has been proven that the _____ property and the _____ property both apply to multiplication.
(commutative; associative)

28. To multiply a number by $33\frac{1}{3}$, multiply it by ____ and then divide the product you obtain by ____.
(100; 3)

29. The most common way to check multiplication is to _____ multiplier and multiplicand.
(transpose or interchange)

Division

18. Of the four basic operations, division is the one disliked by most people. We hope that these articles on division will help you overcome any weakness in division that you might have. If you dislike division, perhaps developing a skill in doing it will lessen your dislike. We will show you that many difficult-looking problems can be solved quite simply and quickly. When you are finished, maybe you will consider division to be as simple as multiplication.

As you learned before, *division* is the process of finding how many times one number is contained in another. Division is the reverse of multiplication. It is the process of separating a number into equal parts or factors. The number to be divided is called the *dividend,* and the number by which we divide is called the *divisor.* The complete result of division is called the *quotient.* If the division is not exact, the part of the dividend that is left over is called the *remainder.* The symbol ÷ is called the *division sign.* It is read "divided by"; and when it is used, therefore, the divisor must *follow it.* For example, 2562 ÷ 7 is read 2562 divided by 7. Here 2562 is the dividend and 7 is the divisor. The quotient in this case is

366. From this it can be seen that the factors of 2562 are 7 and 366 or the prime factors are 2, 3, 7, and 61.

Shortcuts in Division

19. Actually, there are few shortcuts for division. At best it is a long tedious process. However, since division requires skill in multiplication and subtraction, your skill in division will depend to a large degree upon how well you can multiply and subtract. Remember that division is the inverse, or opposite, of multiplication. However, unlike multiplication, division does not possess the commutative property, simply because $15 \div 5$ and $5 \div 15$ are not the same. Nor does division possess the associative property. Try $(45 \div 15) \div 5$ and $45 \div (15 \div 5)$. In the first illustration you get $\frac{3}{5}$; while in the second 15 is the correct answer.

Because division does not possess either the commutative or the associative property, the inverses of our first shortcuts for multiplication do not work. However, the fact that division is the opposite of multiplication does suggest one shortcut. You will recall that to multiply by a number ending in zero or zeros, you moved the decimal point in the multiplicand *to the right* as many places as there were zeros in the multiplier. Doing just the opposite of this, or moving the decimal point in the dividend to the left, will be dividing. Thus for division by 10, 100, 1000, and so on the decimal point in the dividend is simply moved as many places to the left as there are zeros in the divisor. As you should realize, we will form a number smaller than the dividend. Thus, $2730 \div 10 = 273$. The following examples illustrate this kind of division:

$375.45 \div 100 = 3.7545$ $\qquad$ $976.1 \div 10 = 97.61$

$6542.1 \div 1000 = 6.5421$ $\qquad$ $86432 \div 1000 = 86.432$

$5.4326 \div 100 = 0.054326$

Now that you can divide quickly and easily by 10, 100, and 1000, you can extend this method to include division by 20, 30, 400, and so forth, just as we did for multiplication. For example, when dividing by 40, which is equal to 4×10, divide first by 10 and then by 4. Here is how an actual division would be done.

$$248 \div 40 = \frac{248}{40} = \frac{248}{10 \times 4} = \frac{24.8}{4} = 6.2$$

First divide 248 by 10 and then divide that result by 4.

Of course, you do not have to indicate all of these steps when using this shortcut. Just move the decimal point and then divide by the other factor. Thus,

$$3327 \div 300 = 33.27 \div 3 = 11.09$$

$$4855 \div 5000 = 4.855 \div 5 = 0.971$$

Other Shortcuts in Division

20. Here is a list of the shortcuts for division that we consider most useful.

Like the shortcuts for multiplication, they can be of help to you. Study each one carefully and use it whenever possible.

a) To divide a number by 50, divide it by 100 and then multiply by 2.
b) To divide a number by 25, divide it by 100 and then multiply by 4.
c) To divide a number by 20, divide it by 100 and then multiply by 5.
d) To divide a number by $33\frac{1}{3}$, divide it by 100 and then multiply by 3.
e) To divide a number by $12\frac{1}{2}$, divide it by 100 and then multiply by 8.
f) To divide a number by $16\frac{2}{3}$, divide it by 100 and then multiply by 6.

Again, *(a)* and *(c)* represent different ways of dividing by 50 and 20. See Arts. 19 and 16.

Ways to Check Division

21. A simple way to check division is to multiply the quotient by the divisor and then add any remainder to that product. This result should equal the dividend if your work is correct. The following examples show how to check division.

EXAMPLE 1.

$$\begin{array}{r} 7 \\ 55\overline{)387} \\ 385 \\ \hline 2 \end{array}$$ remainder

$55 \times 7 = 385$

$+2$

387 dividend

EXAMPLE 2.

$$\begin{array}{r} 27 \\ 438\overline{)11826} \\ 876 \\ \hline 3066 \\ 3066 \\ \hline \end{array}$$

438

27

3066

876

11826 dividend

EXAMPLE 3.

$$\begin{array}{r} 283 \\ 175\overline{)49532} \\ 350 \\ \hline 1453 \\ 1400 \\ \hline 532 \\ 525 \\ \hline 7 \end{array}$$ remainder

283

175

1415

1981

283

49525

7

49532 dividend

Finally, as we have said many times before, your number sense should give you a quick check of the reasonableness of your answer.

		Approximate Answer
EXAMPLE 4.	$687654 \div 567 = 1212.79$	$700{,}000 \div 600 = 1200$
EXAMPLE 5.	$11638 \div 38 = 306.26$	$12{,}000 \div 40 = 300$
EXAMPLE 6.	$49532 \div 175 = 283.04$	$50{,}000 \div 200 = 250$

Check Your Learning

30. Division is a process of separating a number into _____ parts.

(equal)

31. When you divide a number by 10, 100, 1000, and so forth, the decimal point in the dividend is moved as many places to the _____ as there are zeros in the divisor.

(left)

32. To divide a given number by 70 ($7 \times 10 = 70$), you would first divide by _____ and then by _____.

(10; 7 or 7; 10)

33. In dividing by 10, 100, 1000, and so forth, the number you obtain will be (smaller, larger) than the dividend.

(smaller)

34. The most common way to check division is to _____ the quotient by the divisor and add any _____ to that product.

(multiply; remainder)

35. Division (does, does not) possess the commutative or associative property.

(does not)

Practice Problems

1. Perform the indicated multiplication using a short-cut method wherever possible. Be sure to check your work.

   a) 397×100
   b) 973×800
   c) 45×306
   d) $30 \times 20 \times 50$
   e) $42 \times 37 \times 23$
   f) 253×89

2. Perform the multiplications in the following numbers. Then check your work by approximation.

   a) 229×726
   b) 693×83
   c) 910×456
   d) 38×843
   e) 409×89
   f) 58×719

3. Find the following quotients. If the division is not exact, round off your result to one decimal place. In problems *(a)*, *(b)*, and *(c)* check your work by approximation.

   a) $200\overline{)5600}$
   b) $26\overline{)8918}$
   c) $54\overline{)349272}$
   d) $24\overline{)58819}$
   e) $478\overline{)65109}$
   f) $700\overline{)51406}$

4. If there are 179 days in the average school year, what is the total number of days a student attends the elementary and secondary schools (12 years) when his attendance is perfect?

5. Joan's mother bought a washing machine costing $285. If she paid $45 in cash and arranged to pay off the balance in 8 equal monthly payments, what is the amount of each payment?

Why We Need Better Methods of Checking

22. Throughout this text we have stressed the need for checking your work to make sure your answers are correct. In mathematics there is absolutely no excuse for a wrong answer. The methods of checking addition and multiplication previously given have a common weakness. This weakness is the unpleasant fact that if you made an error in adding or multiplying two numbers, you will perhaps make the same error when you add or multiply them again even though you do reverse the numbers. The checks for subtraction and division do not possess this weakness, and so they are more reliable. The check for subtraction is an especially good one, but the check for division is neither quick nor easy. Hence, it too leaves something to be desired. These are some of the reasons why we are going to learn two other methods of checking: casting out nines and casting out elevens. These methods are based upon the same general principle and are very similar. In both, you work with "check figures" that are completely different from the figures in the problem. Therefore, repeating the same error is less likely to occur. Either method alone provides a good checking device; but when the two are used together, they are just about foolproof.

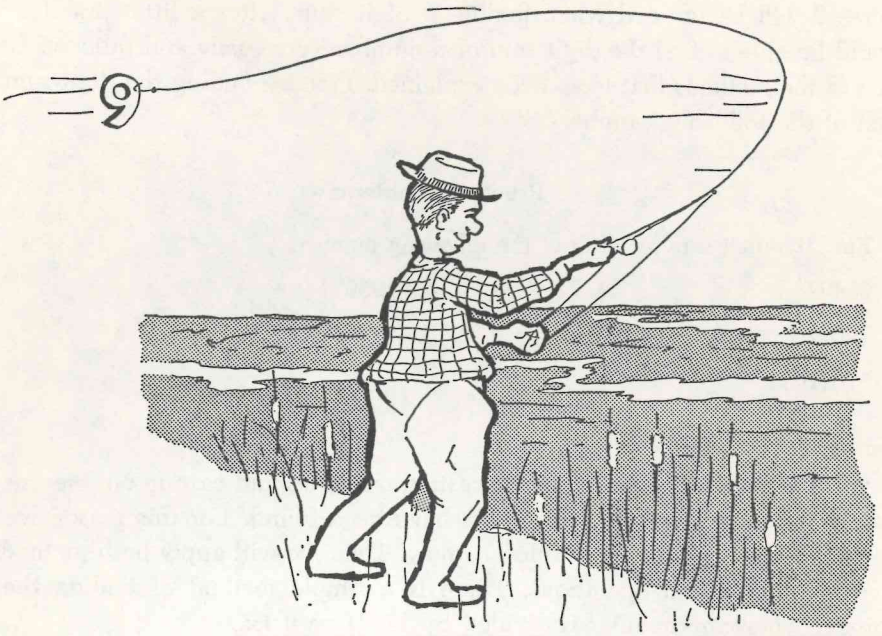
Why These Checks Work

23. The idea upon which these methods is based is a simple one. It is this: *Using the same divisor we divide each of the numbers involved in any operation of arithmetic, and we use the remainders to repeat the same operation and to check our results.* This idea is not as odd as it may seem at first. If you add a series of numbers that are exactly divisible by 5, their total must be divisible by 5 also. Furthermore, if one of the numbers had a remainder of 3, the answer would also have a remainder of 3. This same reasoning can be applied to each of the other operations. For example, if you multiply some numbers that are exactly divisible by 7, their product would necessarily be divisible by 7. Whenever numbers are not exactly divisible by the divisor, then the remainder of each number carries through the operations. For this reason, when you do to these remainders what you did to the numbers themselves, your result will equal the remainder of the answer.

Casting Out Nines

24. If we had to actually perform the divisions suggested in the preceding article, the new methods of checking would certainly not be easy or practical. But out of all the numbers that could be used as divisors, there are two that have a most unusual characteristic. These numbers are 9 and 11, and they are unique because it is very simple to calculate the remainder when any number is divided by either of them. First we will discuss dividing by 9, which mathematicians call casting out nines.

When you divide a number by 9, the remainder is the same as the sum of the digits of the number. If this sum has more than one digit, you find the sum of its



digits. You continue to do this until you obtain a sum consisting of one digit only. For example, to find the remainder when 37463 is divided by 9, you add the digits; thus $3 + 7 + 4 + 6 + 3 = 23$. To reduce 23 to a single digit, add its digits 2 and 3 and obtain the sum of 5. If you divide 37463 by 9, the remainder will be 5. Try it.

Because the sum of the digits or the digit sum gives us the remainder when we use 9 as the divisor, we will study some of the "tricks" in obtaining the digit sum.

The Digit Sum

25. The digit sum of any number can be found by the method explained in the preceding article. However, there are many little tricks that make it easier and quicker to find the digit sum. In the number 37463, which we used before, you can find the digit sum this way: 3 plus 7 is 10. Reduce this by adding 1 and 0 to get 1, then add 1 to 4 (the next digit in the number) to get 5; $5 + 6 = 11$. Reducing again, $1 + 1 = 2$; and 2 plus $3 = 5$. This is the same answer as before.

Another way to find the digit sum is to proceed as follows: Again we will use 37463. Adding the first two digits, $3 + 7 = 10$, subtract 9 (cast out nine) and you get 1. Now. $1 + 4 = 5$ and adding 6 gives 11. Casting out nine gives 2; then adding 3 gives 5. The fact that you can cast out nines simplifies finding the digit sum greatly, because now when you see a 9 or several numbers that add up to 9 you can ignore them. Therefore when finding a digit sum, you can glance over the digits in the number and "cast out" any that are 9 or that add up to nine. Of

course 0 will be ignored when finding a digit sum. After a little practice you should be able to find the digit sum of a number very easily and quickly. Learn each of the methods that have been explained. Practice finding the digit sum for each of the following numbers.

Practice Problems

Find the digit sum for each of the following numbers.

1. 65432
2. 79672
3. 5767231
4. 125971
5. 876347

Casting Out Elevens

26. The methods of checking by casting out nines and casting out elevens are exactly the same after the remainders have been found. For this reason we are going to learn how to cast out elevens now. Then we will apply both methods in checking the several operations. There is a simple method of finding the remainder when any number is divided by 11. Here it is:

Start with the first digit (the digit on the left); subtract it from the second digit; subtract this result from the third digit; and so forth. For example, to find the remainder when 34783 is divided by 11, these are the mental calculations:

$$4-3=1 \qquad 7-1=6 \qquad 8-6=2 \qquad 3-2=1. \quad \text{Ans.}$$

Hence 1 is the remainder when 34783 is divided by 11. If any digit is too large to be subtracted from the next digit, add 11 to that digit and then perform the subtraction.

EXAMPLE 1. Find the remainder when 4321 is divided by 11.

SOLUTION. Here $3 - 4$ is not possible, so we add 11 to 3 and get 14. Then $14 - 4 = 10$. Again we add 11 to the next digit 2, which gives 13. $13 - 10 = 3$, and 12 (which is $11 + 1$) minus 3 gives 9, which is the remainder. Check this answer by doing the actual division.

EXAMPLE 2. Find the remainder when 783562 is divided by 11.

SOLUTION. Very briefly this is the way we should think: "7 from 8 is 1 from 3 is 2 from 5 is 3 from 6 is 3 from 13 is 10." Thus 10 is the remainder.

Use the following to practice casting out elevens.

Practice Problems

Cast out elevens in the following numbers:

1. 9714376
2. 5321675
3. 3198765
4. 1763428
5. 24345326

Checking by Casting Out

27. Now that you can calculate the remainder when a number is divided by either 9 or 11, we will show you how to check addition, multiplication, and division. There will be no check shown for subtraction, because as we have said before, the check given previously for subtraction is an excellent one. For each operation we will give both checks at the same time. You should do this too when you check your work.

a) Addition. To check addition, find the remainders (by casting out nines and by casting out elevens) for each of the addends and then add the remainders. The sum of the remainders should equal the remainder of the sum.

		Remainders	
		Casting Out 9	Casting Out 11
EXAMPLE 1.	4723	7	4
	1465	7	2
	326	2	7
	15	6	4
	6529	$22 = 4$	$17 = 6$

The remainder for 6529 by casting out nines is 4, and the remainder by casting out elevens is 6. Therefore the answer is correct.

b) Multiplication. To check multiplication, find the remainders (9's and 11's) for each of the factors. Multiply the remainders, and the product should equal the remainders of the product.

			Remainders	
			Casting Out 9	Casting Out 11
EXAMPLE 2.	327	multiplicand	3	8
	214	multiplier	7	5
	69978	product	$21 = 3$	$40 = 7$

The remainders for 69978 by the two methods are 3 and 7. This confirms our result.

c) Division. When checking division, we must change our procedure slightly. If you were to try dividing the remainders of the dividend by the remainders of the divisor, you would encounter difficulties that can be avoided by the following method.

To check division, whether it is by casting out nines or by casting out elevens, multiply the remainder of the quotient by the remainder of the divisor. If the division is exact, this result should equal the remainder of the dividend. However, if the division is not exact you must add either the remainder of the division itself or its remainder after casting out, to the product of the remainders of the quotient and the divisor. If your work is correct this sum should then equal the remainder of the dividend.

EXAMPLE 3.

```
      231
357)82467
  714
  1106
  1071
    357
    357
```

Remainders

Casting Out 9	Casting Out 11
divisor 6	5
quotient 6	0
$36 = 0$	0

Here the division is exact and the remainders of the dividend by the two methods are 0 and 0. Therefore the division is correct.

EXAMPLE 4.

```
     58
47)2730
  235
  380
  376
    4  remainder
```

Remainders

Casting Out 9	Casting Out 11
divisor 2	3
quotient 4	3
8	9
$4 + 8 = 12 = 3$	$4 + 9 = 13 = 2$

Here the division is not exact and the remainder of the division, 4, must be added to the product of the remainders of divisor and quotient. The results check the remainders of the dividend, 3 and 2.

EXAMPLE 5.

```
     327
426)139422
  1278
   1162
    852
    3102
    2982
     120  remainder
```

Remainders

Casting Out 9	Casting Out 11
divisor 3	8
quotient 3	8
$9 = 0$	$64 = 9$
$3 + 0 = 3$	$10 + 9 = 19 = 8$

Here the division is not exact and the remainder of the division, 120, has remainders of its own: 3 and 10. The procedure is then the same as in Example 4. The remainders of the dividend are 3 and 8, which check.

Check Your Learning

36. The methods of checking by casting out nines and casting out elevens are based upon the same general _____ and are very similar.

(principle)

37. For these methods you use the same _____ to divide each of the numbers involved in any operation of arithmetic.

(divisor)

38. When the division is performed use the _____ to repeat the same operation and to check your results.

(remainders)

39. If you multiply some numbers that are exactly divisible by 7, their _____ would necessarily be divisible by 7.

(product)

40. Whenever numbers are not exactly divisible by the divisor, then the _____ of each number carries through the operations.

(remainder)

41. When you divide a number by nine, the remainder is the same as the _____ of the digits of the numbers.

(sum)

42. The digit sum when casting out nines (can, cannot) contain more than one digit.

(cannot)

43. When finding a remainder by casting out elevens, if any digit is too large to be subtracted from the next digit, add _____ to that digit and then perform the subtraction.

(11)

Practice Problems

After solving the following problems check your work by casting out nines and by casting out elevens.

1. Add:

a)	*b)*	*c)*	*d)*	*e)*
5726	767	62	5767	14132
357	1511	375	9708	563
13056	96	1111	965	7964
21	384	81	3654	866

2. Find the product of:
   - *a)* 976 and 54
   - *b)* 1575 and 2804
   - *c)* 361 and 542
   - *d)* 767 and 810
   - *e)* 511 and 343

3. Divide the following:
   - *a)* 7634 by 57
   - *b)* 57674 by 707
   - *c)* 769546 by 222
   - *d)* 967548 by 345
   - *e)* 197076 by 67

Grouping Symbols

28. In some problems it will be necessary to perform certain operations first and to combine certain numbers. Suppose you were asked to write, with numbers and symbols, that the sum of 12 and 21 is to be subtracted from 97 and that this result is to be divided by the sum of 5 and 3. How can you show clearly the numbers which are to be grouped together and the order in which the operations are to be performed? By using symbols of aggregation, problems such as this one can be written quite clearly. The symbols of aggregation are sometimes called symbols of grouping, and you have already used two such symbols in your work

thus far. These symbols are the bar, sometimes called the vinculum, and the parentheses, ( ). The bar in a fraction tells you that the entire numerator is divided by the entire denominator. The parentheses are used to group numbers together so that they will be considered as a single number. There are two other sets of symbols: the brackets, [ ], and the braces, { }. Any of these symbols may be used for grouping; but when only one set is needed, parentheses are generally used. When more than one set of grouping symbols is necessary, it is best to use different symbols.

Use of Grouping Symbols

29. Let's go back to our problem and write it by using grouping symbols. The sum of 12 and 21 is written as $12 + 21$, but when this sum must be considered as a single number to be subtracted from 97, it is indicated in this manner:

$$97 - (12 + 21)$$

The sum of 5 and 3 is $5 + 3$, but when you divide $97 - (12 + 21)$ by this sum, you must enclose $5 + 3$ within a symbol of grouping. The entire expression becomes $[97 - (12 + 21)] \div (5 + 3)$. Can you see why we used brackets? It is because 97 minus the sum of 12 and 21 is to be divided by the sum of 5 and 3. Therefore, a grouping of $97 - (12 + 21)$ was necessary, and the brackets were used because parentheses had been used to group 12 and 21. Obviously, all of the numbers within each symbol must be simplified into a single number before either the subtraction or the division can be performed. Whenever there are symbols of grouping, you must perform the operations indicated within each set before continuing. Here are the steps in simplifying the expression we have been talking about:

$$[97 - (12 + 21)] \div (5 + 3)$$
$$[97 - 33] \div 8$$
$$64 \div 8 = 8 \quad \text{Ans.}$$

Notice that when the numbers within each set of grouping symbols were expressed as a single number, the symbol was dropped. We did this because the symbol was no longer necessary. Surely you know that this problem could have been written in another way. Namely

$$\frac{97 - (12 + 21)}{5 + 3}$$

However, in this case, we are still using two sets of grouping symbols: the fraction bar and the parentheses.

Here are the important rules to keep in mind.

1. Use symbols of grouping to write a series of operations clearly and correctly — never use them unless they are needed for clarity and correctness.

2. Always simplify the numbers within any set of symbols into a single number before working any further with the problem.
3. When symbols of grouping are used within other symbols, you should work with the innermost symbol first and then continue until all symbols are removed.

Study the following example. For the first step we simplified $9 + 7 - 12$ in accordance with rule 3:

$$[25 - (9 + 7 - 12)] + (3 + 4) \times (1 + 7)$$
$$[25 - (4)] + (7) \times (8)$$
$$[25 - 4] + 56$$
$$21 + 56 = 77 \quad \text{Ans.}$$

Here are some problems for your practice.

Practice Problems

Remove the symbols of aggregation and simplify.

1. $\dfrac{56 + (48 - 32)}{4 \times 2}$
2. $[45 - (5 + 3 + 7)] \div (3 \times 5)$
3. $45 \div (10 + 5) + 5 \times 4 - (8 + 2 - 4)$
4. $[(3 + 5) \times (19 - 9)] \div 10 + (15 - 2 - 1) \div 4$
5. $25 + (6 - 5) - (3 + 8)$

Use of Grouping Symbols for Multiplication

30. Now that you know how symbols of grouping can be used and how to work with them, there is another important use which you should know about. In the past we have indicated multiplication by the times sign, $\times$. There is another way to show multiplication. Symbols of grouping are often used to indicate this operation. Whenever there is no sign between any grouping symbol and a number or between sets of grouping symbols, multiplication is indicated and must be performed. For example, $3(2 + 5)$ and $(2 + 7)(3 - 1)$ mean 3×7 and 9×2. Of course, as before, you simplify within the symbols before performing the multiplication. Consider the problem of finding the value of $5(7 + 2 - 3) - 7$. First, simplify within the parentheses like this: $7 + 2 - 3 = 9 - 3 = 6$. Now you have $5(6) - 7$, which becomes $30 - 7 = 23$, the answer. Take the expression $(4 + 10)(3 + 8)$, which has two sets of symbols used to indicate multiplication. Proceed in the same way you have been. Thus, $4 + 10$ is 14, $(3 + 8)$ is 11, and $(4 + 10)(3 + 8)$ becomes $14 \times 11 = 154$, the answer.

Distributive Property of Multiplication

31. When you were looking at the problems $3(2 + 5)$ and $5(7 + 2 - 3) - 7$,

the thought may have occurred to you that multiplication could have been performed without first simplifying within the parentheses. Perhaps you weren't certain about whether it was correct, so you dismissed the idea without even trying to see if you could get the correct answer by using this method. The truth of the matter is that multiplication can be *distributed* over addition and subtraction because of the principle known as the *distributive property of multiplication.* This last statement requires an explanation, and the best way to illustrate this new property is by showing examples. We'll use the two problems given above.

You already know that $3(2 + 5)$ is equal to $3(7) = 21$, but now by applying the distributive property of multiplication we can find this product in another way. Thus,

$$\begin{aligned} 3(2+5) &= 3 \times 2 + 3 \times 5 \\ &= 6 + 15 \\ &= 21 \quad \text{Ans.} \end{aligned}$$

The distributive property involves nothing more than the multiplication of every number within the symbol by the multiplier. Whether you simplify within the symbols first or whether you use the distributive property, the result will be the same. Now we'll do the other problem.

EXAMPLE. Simplify $5(7 + 2 - 3) - 7$.

SOLUTION. 1. Simplify first within the parentheses:

$$\begin{aligned} 5(9-3) - 7 \\ 5(6) - 7 \\ 30 - 7 = 23 \quad \text{Ans.} \end{aligned}$$

SOLUTION 2.

$$\begin{aligned} 5(7+2-3) - 7 \\ 5 \times 7 + 5 \times 2 - 5 \times 3 - 7 \\ 35 + 10 - 15 - 7 \\ 45 - 15 - 7 \\ 30 - 7 = 23 \quad \text{Ans.} \end{aligned}$$

Obviously, this distribution of the multiplication doesn't alter the correct result and can be used to simplify expressions such as these. The distributive property of multiplication is discussed in the new modern mathematics approach, but again it is a property which is really not "new." If you find the distributive property convenient to use, be sure that you keep in mind that multiplication can be distributed over addition and subtraction only.

Using the distributive property to simplify $28 + 2(6 + 9 - 4)$ gives a solution as follows:

$$\begin{aligned} 28 + 2(6+9-4) \\ 28 + 2 \times 6 + 2 \times 9 - 2 \times 4 \\ 28 + 12 + 18 - 8 = 58 - 8 = 50 \quad \text{Ans.} \end{aligned}$$

There are times when you'll want to use the distributive property, but be careful to apply it correctly. Be sure that you multiply *every* number within the symbol connected by a plus or minus sign.

Practice Problems

Solve the following problems.

1. $23 - (3 + 2)\ (8 - 6)$
2. $2 + 6(2 + 1) + (5 + 4 - 3)7$
3. $(6 + 2)9 - 60 + 3$
4. $27 - 3 - 2(7 + 2)$
5. $(8 - 3)4 + (6 + 1)\ (5 - 2)$

Order of Operations

32. You have learned that a problem can involve more than one operation and that symbols of grouping are used to indicate the proper procedure for solving such a problem. However, you will also come across problems in which all four basic operations are indicated and must be performed in which there are sometimes no symbols to show the way to the correct solution. What do you do about solving problems of this kind? Once again, there is a rule to follow. It is known as the *order of operations.* As you will recall, you were introduced to this principle in *Practical Arithmetic,* Part 2. In this article, the order of operations will be extended so that you can solve problems a little more difficult than the ones you have done before.

We'll begin by stating the rule for the order of operations.

RULE. *In a series of operations perform all of the multiplications and divisions in the order in which they appear from left to right. Then perform all of the additions and subtractions.*

If this rule is followed, there can be only one correct answer. Mathematicians have agreed on this rule; therefore, there is only one correct way to simplify an expression when several operations must be done.

When you are presented with a problem like $7 + 14 \div 7 - 4 \div 2 + 3 \times 7$, there should be absolutely no doubt in your mind as to the correct procedure. Don't let anyone confuse you by arguing that there are several ways to solve this problem and several answers any one of which is correct. There is *only one* correct procedure and *only one* correct answer. Keep the rule in mind and you won't go wrong. Now we'll solve this problem.

$$7 + 14 \div 7 - 4 \div 2 + 3 \times 7$$
$$7 + 2 - 2 + 21$$
$$9 - 2 + 21 = 7 + 21 = 28 \quad \text{Ans.}$$

Do you see how simple the problem becomes when you just follow the rule? Let's try another problem:

$$24 \div 6 + 3 \times 4 - 16 \div 8 + 10 + 3 - 2$$
$$4 + 12 - 2 + 10 + 3 - 2$$
$$16 - 2 + 13 - 2$$
$$14 + 11 = 25 \quad \text{Ans.}$$

Suppose that symbols of grouping as well as several other arithmetic operations are given in the problem. What should you do then? Remember to remove the symbols of aggregation first and then follow the order of operations. For example,

$$4 \times 10 + 15 - 18 \div 2 - [6 + 2(7 - 5)]$$
$$4 \times 10 + 15 - 18 \div 2 - [6 + 2(2)]$$
$$4 \times 10 + 15 - 18 \div 2 - [6 + 4]$$
$$4 \times 10 + 15 - 18 \div 2 - 10$$
$$40 + 15 - 9 - 10$$
$$55 - 9 - 10 = 46 - 10 = 36 \quad \text{Ans.}$$

Check Your Learning

44. Symbols of aggregation are sometimes called symbols of _____.
(grouping)

45. To indicate which operations are to be performed first we can use _____ _____ _____.
(symbols of aggregation)

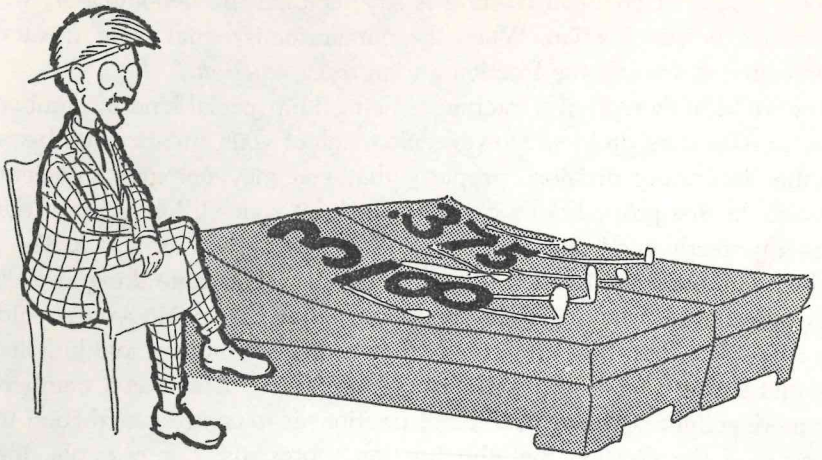
46. When only one set of symbols is necessary in a problem, the _____ are generally used.
(parentheses)

47. If there isn't a sign written between parentheses and a number the operation of _____ is indicated.
(multiplication)

48. To find the value of $9(6 + 12)$, we can add first and then multiply, or we can use the _____ _____ to distribute the multiplication over the addition.
(distributive property)

49. The principle of Order of Operations tells us that we must multiply and divide (before, after) we add and subtract.
(before)

50. The correct value of $(18 + 10 \times 3 \div 5)2 - 4 \times 3$ is _____.
(36)



Practice Problems

Solve the following:

1. $3 + 5 \times 2 + 8 \div 2 - 1$
2. $12 + 8 \div 4 + 15$
3. $3 \times 6 \div 3 - 3 \times 2$
4. $\dfrac{15 \div 5 \times 3}{27 \div 9}$
5. $24 \div 6 \times 2 + 10 - 3$

A Better Understanding of Fractions and Decimals

The Purpose of This Section

33. In the next few articles we are going to review the high points of calculations involving fractions and decimals. We will touch upon some of the things that are troublesome for some people. And finally we will show how modern mathematics has explained some of the things that previously were taken for granted. Before you continue beyond this section, be sure that your knowledge of both fractions and decimals is the very best. If you realize that your knowledge is weak on some points, go back and review or restudy the appropriate topics in the preceding texts.

Fractions

34. Every fraction is composed of two parts. One part, the numerator, is written above the fraction bar. The other part, the denominator, is written under the fraction bar. These parts are called the *terms* of the fraction. You have learned also that either or both terms can be fractions or expressions involving the four basic operations (addition, subtraction, multiplication, and division). When the

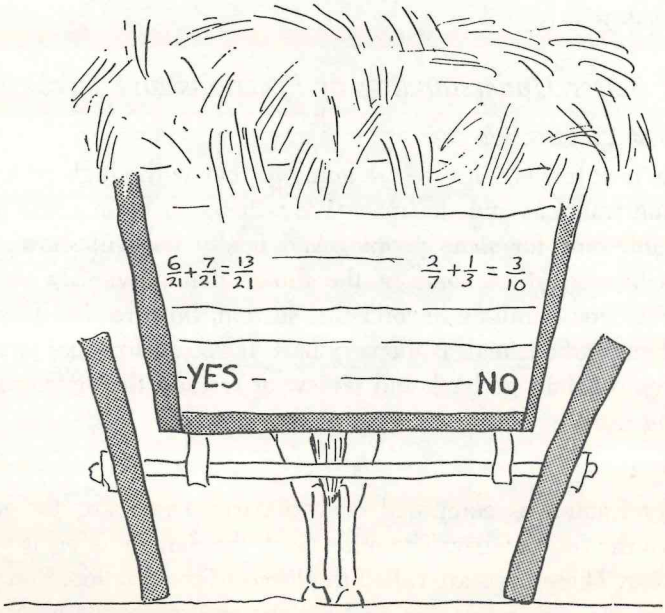
numerator of a simple or common fraction is smaller than the denominator, we call the fraction a *proper fraction.* When the numerator is equal to or greater than the denominator, we call the fraction an *improper fraction.*

Up to now we have thought of a fraction as being 1) a special kind of number and 2) a way of indicating division. However, so much of your attention has been devoted to the "indicating division" property that you may not give sufficient consideration to the first property of a fraction. Don't let yourself lose sight of the fact that $\frac{1}{2}$ is a perfectly good number; it is just halfway between 0 and 1. True, $\frac{1}{2}$ is not a whole number or an integer, but it is a number. The same thing applies to all fractions – they are numbers! If you think about this point, you should come to the conclusion that there are literally thousands, yes, millions and billions, of fractions that lie between 0 and 1. Every one of these fractions is a number!

Another more enlightened way to think of fractions is to consider a fraction to be the quotient of the division that the fraction represents. For example, the fraction $\frac{3}{4}$ can be thought of as 1) a number, 2) a problem in division $(3 \div 4)$, and 3) a quotient (the result of dividing 3 by 4). Don't think, as some persons do, that the quotient of $3 \div 4$ can be expressed only as 0.75!

The Basic Operations for Fractions

35. In adding or subtracting fractions you must be sure that their denominators are the same number. If they are not, find the least common denominator and rewrite each fraction with this denominator. For addition, add the numerators; and for subtraction, subtract the numerators. In each case the denominator



of the answer will be the LCD (least common denominator). *Never add or subtract the denominators!*

For multiplying fractions the procedure is very simple. You multiply the numerators together to get the numerator of the product and multiply the denominators together to get the denominator of the product. For division, however, you will recall that no specific method of dividing was given. Instead, you were taught to convert all divisions involving fractions into multiplications of fractions. This is really a simplification, because you have to learn only one operation: multiplication. We're sure you remember the rule for dividing fractions. It is:

RULE. *To divide fractions, invert the divisor and multiply.*

This rule is true because division is the *inverse of multiplication*. You will recall, too, that it is possible to simplify or to speed up both multiplication and division by canceling whenever possible. Finally, don't forget that for all of your work in fractions you should express your answer in lowest terms.

Some Warnings About Fractions

36. Many people have trouble with fractions. Often the cause of their difficulties is a misunderstanding of some very minor point rather than not knowing a major principle. Sometimes in your hurry to get things done you will skip over a simple but important operation. Here are some things you should keep in mind at all times. All of them have been mentioned before.

1. Don't add denominators when adding fractions.
2. Don't forget to invert the divisor when dividing fractions. Never invert the dividend! At this point, it is appropriate to note that a whole number can be written as a fraction merely by writing it over 1. Thus, $15 = \frac{15}{1}$. Hence when you invert $\frac{15}{1}$, you get $\frac{1}{15}$. This point comes up in a problem like this:
$$\frac{3}{4} \div 15 = \frac{3}{4} \times \frac{1}{15} = \frac{1}{20} \quad \text{Ans.}$$
3. Don't put the dividend after the sign of divison. This is a very common error. For example, $\frac{2}{5}$ divided by $\frac{2}{3}$ is not $\frac{2}{3} \div \frac{2}{5}$.
4. Always express any fractional answer as a fraction in its lowest terms. In most cases, express an improper fraction as an integer or as a mixed number.

Fundamental Law of Fractions

37. In *Fractions* you noticed that the following rule was given many times: You can multiply or divide both the numerator and denominator of a fraction by the same number without changing the value of the fraction. This rule is so important in mathematics that we are going to call it the *fundamental law of fractions*. The usefulness of the principle stated by this rule can be seen in practically all work that involves fractions, and this usefulness is not limited to fractions in arithmetic! You will find this principle working throughout all mathe-

matics, and, unfortunately, it causes a great deal of trouble because people ignore it. You used this principle when you wrote fractions so that they had a different denominator. For example, to change $\frac{3}{4}$ to a fraction having a denominator of 12, you proceeded like this:

$$\frac{3 \times 3}{4 \times 3} = \frac{9}{12}$$

You used it when you reduced fractions to lower terms. For example,

$$\frac{24 \div 8}{32 \div 8} = \frac{3}{4}$$

Later on you will meet other applications of this same rule.

The fundamental law of fractions embodies a most important principle, but our reason for mentioning it at this time is to show you why it is true. Everyone knows that the product of 1 and any number is the number itself. Likewise, the quotient of any number divided by 1 is the number itself. In other words, multiplying or dividing a number by 1 does not change the number. Recall also that the number 1 can be written in many, many ways, such as

$$\frac{2}{2} \qquad \frac{3.5}{3.5} \qquad \frac{\frac{1}{2}}{\frac{1}{2}} \qquad \frac{(2+3)}{(4+1)}$$

and so on. These are the facts that justify the fundamental law. Notice that when you multiply both numerator and denominator by the same number, you are actually multiplying by 1. Likewise, when you divide both numerator and denominator by the same number, you are dividing by 1. Therefore, there is no change in the value of the fraction!

Check Your Learning

51. The numerator and denominator are called _____ of a fraction.

(terms)

52. Every fraction (is, is not) a number.

(is)

53. In adding or subtracting fractions, the denominators (should, should not) be added or subtracted.

(should not)

54. The fundamental law of fractions states that you can _____ or _____ both the numerator and denominator of a fraction by the same number without changing the value of the fraction.

(multiply; divide)

55. When you multiply or divide a number by 1 it (does, does not) change the number.

(does not)

The Identity Property

38. The property of the number 1, namely, that 1 can be used as a multiplier or divisor and no change will occur, is a very unusual or unique property. We call the property unique because it is possessed by no other number. Here is another topic upon which modern mathematics has concentrated considerable attention. In fact, a specific name has been given the number 1. It is called *the identity element for multiplication.* This name is based upon the fact that either 1×5 or 5×1 is 5. That you can divide by 1 without change does not warrant calling 1 an identity element for division, because $4 \div 1$ and $1 \div 4$ are not the same. It is to matters such as these that modern mathematics devotes much of its attention.

The point we want to make, and it is a very important point, is this: You should understand why the fundamental law of fractions is true and how it can be used to write a fraction that is equal to another fraction. Such equal fractions are sometimes called *equivalent fractions.*

Decimals

39. In *Practical Arithmetic,* Part 3, you'll find a definition of a decimal. It reads like this: *A decimal is a fraction with an unwritten denominator, preceded by a decimal point whose position indicates the value of the denominator.* You recall, of course, that in *Decimals* we said also that 0.36, 0.0215, and 0.374 are decimals and numbers such as 1.25, 31.62, and 12.75 are called mixed decimals. However, at this point we want you to think of a decimal not as something that is very different from whole numbers and fractions, but as an extension of our number system. This idea should not be too difficult for you to understand, because you should be accustomed to handling decimals and mixed decimals just as easily as you handle integers. Decimals, as such, are actually a continuation, or refinement, of our system of whole numbers, and the basic operations for them are the same as they are for the integers. We need no new rules for decimals as we needed them for fractions. This fact alone makes it more convenient to work with decimals than it is to work with fractions.

The Basic Operations for Decimals

40. Because all the rules for integers apply to decimals, and here we mean decimals and mixed decimals as well, it is a simple matter to review the basic operations. For addition and subtraction, the important thing to remember is to *line up the decimal points,* then you proceed just as you would for whole numbers. However, keeping the decimal points in the same column seems to be the hardest thing for some people to do. So be sure you don't make this very common error.

For multiplication of decimals, you do exactly as you would do for integers. But don't forget to point off your answer. To do this correctly all you have to remember is the following rule.

RULE: *There are as many decimal places in a product as there are decimal places in both factors.*

In the division of decimals there are several things that you must keep in mind, but they are simple things.

1. *When the divisor is an integer, the decimal point in the quotient is directly over the decimal point in the dividend.*
2. *When the divisor is not an integer, we make it an integer by moving the decimal point to the right; at the same time, we move the decimal point in the dividend the same number of places to the right.*

From this point, the division of decimals is exactly like division for whole numbers.

Per Cent

41. No discussion of decimals would be complete if an explanation of per cent was not included. Per cent is really a special kind of decimal. If you always think of per cent in this light, your understanding and ability to handle it will be improved. "Per cent" is an abbreviation of the Latin phrase *per centum*. But even though *per centum* literally means "by the hundred" or "in the hundred," "per cent" means "hundredths." For example, 20 per cent means 20 hundredths, and so it may be written 0.20. It may also be written $\frac{20}{100}$, which reduces to $\frac{1}{5}$. Similarly, $16\frac{2}{3}$ per cent means 0.1667 or, writing it as a fraction, $\frac{16\frac{2}{3}}{100}$. This fraction reduces to

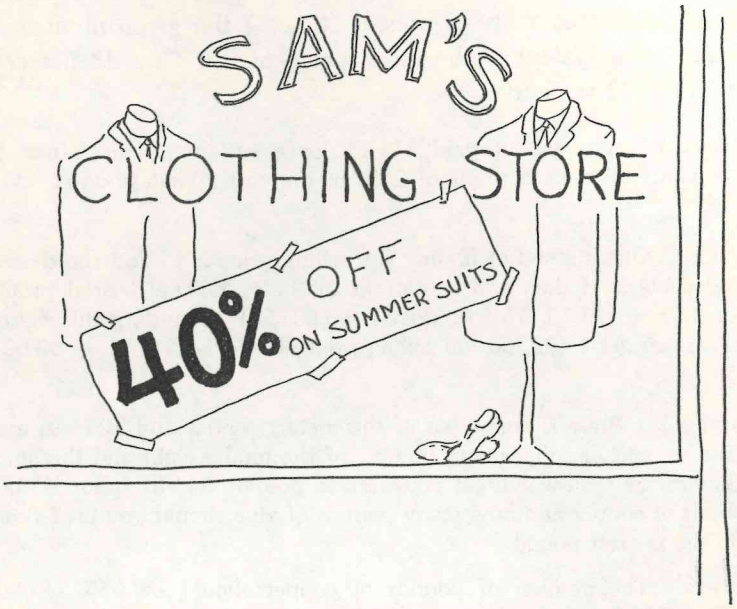
$$\frac{\frac{48+2}{3}}{100} = \frac{\frac{50}{3}}{100} = \frac{50}{300} = \frac{1}{6}$$

A per cent is not always a whole number. Also, a per cent may be a number less than 1, as, for example, $\frac{1}{4}$ per cent. Or it may be a number more than 100, as 150 per cent.

The symbol % is often used instead of the term "per cent." Thus 2% is another way of writing 2 per cent, and 15.4% is another way of writing 15.4 per cent.

Uses of Per Cent

42. Instead of saying that 20 people out of 100 do a certain thing, it is more usual to say that 20 per cent of the people do a certain thing. Also, it is more usual to say that 5 per cent of the pieces produced by a machine were defective than it is to say that 5 pieces out of each 100 pieces produced by the machine were defective. However, when the term "per cent" is used, it is not necessary to refer to exactly 100 people, or 100 pieces, or 100 things of any kind. Any number



of people or things may be considered in connection with per cent, as you will see in the examples and problems that follow.

When you use a per cent in a calculation, replace it by the equivalent decimal. To find the equivalent decimal, divide the number preceding the symbol % or the term "per cent" by 100. You can perform this division simply by moving the decimal point in the number two places to the left. Therefore, to replace a per cent by a decimal, drop the term "per cent" or the symbol % and move the decimal point two places to the left. For example, 10 per cent, or 10%, becomes 0.10; 15.4 percent becomes 0.154; 125 percent becomes 1.25; and 0.3 per cent becomes 0.003.

When you move the decimal point, you must drop the term "per cent" or the symbol %. Note that in the preceding examples we wrote 0.10, 0.154, 1.25, and 0.003 without the term per cent or the symbol %.

In some circumstances you will want to change a decimal to a per cent. To do this, you multiply the decimal by 100 (by moving the decimal point two places to the right) and put the term "per cent" or the symbol % after the result. For example, 0.762 is equal to 76.2 per cent, or 76.2%; and 0.025 is equal to 2.5 per cent, or 2.5%. When you change a decimal to a per cent, use the term "per cent" or the symbol % after the new number.

Problems Involving Per Cent

43. A common type of problem is to find a number that is a certain per cent of another number. For example, you may be required to find the number that is

18 per cent of 62. In a problem of this kind, you find the required number by multiplying (remember that "of" means "times") the given number, 62, by the decimal that is equivalent to the given per cent, 18. Thus 18 per cent of 62 is equal to $0.18 \times 62 = 11.16$.

EXAMPLE 1. A merchant paid \$14.75 for a certain article. When he sells the article, he wants to make a profit of 25% of the cost. What price should he ask for the article?

SOLUTION. One method of finding the selling price is to find the desired profit, in dollars and cents, and then add this profit to the cost. The desired profit is 25% of \$14.75, or $0.25 \times \$14.75$. This product is \$3.6875. By rounding this figure off to the nearest cent, you get \$3.69. So the selling price should be \$14.75 + \$3.69, or \$18.44. Ans.

EXAMPLE 2. Brass is composed of the metals copper and zinc. In a certain kind of brass the amount of copper is to be 65% of the total weight and the amount of zinc is to be 35% of the total weight. If you want to prepare 348 lb (pounds) of brass, how many pounds of copper and how many pounds of zinc should you use? Round off each amount to the nearest pound.

SOLUTION. The number of pounds of copper should be 65% of 348 lb. Since $348 \times 0.65 = 226.2$, the required weight of copper to the nearest pound is 226 lb. Ans.

Similarly, since $348 \times 0.35 = 121.8$, the required weight of zinc to the nearest pound is 122 lb. Ans.

CHECK. As a check on the work, determine whether the sum of the weights of the copper and zinc is 348 lb, as it should be. Since $226 + 122 = 348$, the results or correct.

There are other types of problems involving per cent. We will study them in the last part of *Practical Arithmetic*.

Decimal and Fractional Equivalents

44. Unfortunately, it is not possible to say that you should always use decimals, because sometimes it is better or easier to use fractions. When fractions such as $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{8}$ are involved, it is a matter of personal preference whether you use the fraction or its decimal equivalent. On the other hand, there are many fractions that you will often meet, such as $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{12}$, that are easier to use as fractions than they are as decimals. Other less frequently occurring fractions such as $\frac{4}{7}$, $\frac{21}{31}$, and $\frac{11}{127}$ are obviously more easily handled than their equivalent decimals. To further complicate the situation, there are many decimals that no one would ever attempt to reduce to fractions. Some common examples are 3.1416, 0.7854, and 0.30103. Don't allow yourself to become bewildered by these statements. Unless you are told to do a problem in a certain way, do it in the easiest manner or by using the method you prefer if there are several ways that are about the same in difficulty.

Fractions and Their Decimal Equivalents

45. In Table 1 we have listed all fractions from $\frac{1}{64}$ to $\frac{64}{64}$ (or 1) and their

TABLE 1

DECIMAL EQUIVALENTS OF 64THS

Fraction	Decimal	Fraction	Decimal	Fraction	Decimal	Fraction	Decimal
$\frac{1}{64}$	0.015625	$\frac{17}{64}$	0.265625	$\frac{33}{64}$	0.515625	$\frac{49}{64}$	0.765625
$\frac{1}{32}$	0.031250	$\frac{9}{32}$	0.281250	$\frac{17}{32}$	0.531250	$\frac{25}{32}$	0.781250
$\frac{3}{64}$	0.046875	$\frac{19}{64}$	0.296875	$\frac{35}{64}$	0.546875	$\frac{51}{64}$	0.796875
$\frac{1}{16}$	0.062500	$\frac{5}{16}$	0.312500	$\frac{9}{16}$	0.562500	$\frac{13}{16}$	0.812500
$\frac{5}{64}$	0.078125	$\frac{21}{64}$	0.328125	$\frac{37}{64}$	0.578125	$\frac{53}{64}$	0.828125
$\frac{3}{32}$	0.093750	$\frac{11}{32}$	0.343750	$\frac{19}{32}$	0.593750	$\frac{27}{32}$	0.843750
$\frac{7}{64}$	0.109375	$\frac{23}{64}$	0.359375	$\frac{39}{64}$	0.609375	$\frac{55}{64}$	0.859375
$\frac{1}{8}$	0.125000	$\frac{3}{8}$	0.375000	$\frac{5}{8}$	0.625000	$\frac{7}{8}$	0.875000
$\frac{9}{64}$	0.140625	$\frac{25}{64}$	0.390625	$\frac{41}{64}$	0.640625	$\frac{57}{64}$	0.890625
$\frac{5}{32}$	0.156250	$\frac{13}{32}$	0.406250	$\frac{21}{32}$	0.656250	$\frac{29}{32}$	0.906250
$\frac{11}{64}$	0.171875	$\frac{27}{64}$	0.421875	$\frac{43}{64}$	0.671875	$\frac{59}{64}$	0.921875
$\frac{3}{16}$	0.187500	$\frac{7}{16}$	0.437500	$\frac{11}{16}$	0.687500	$\frac{15}{16}$	0.937500
$\frac{13}{64}$	0.203125	$\frac{29}{64}$	0.453125	$\frac{45}{64}$	0.703125	$\frac{61}{64}$	0.953125
$\frac{7}{32}$	0.218750	$\frac{15}{32}$	0.468750	$\frac{23}{32}$	0.718750	$\frac{31}{32}$	0.968750
$\frac{15}{64}$	0.234375	$\frac{31}{64}$	0.484375	$\frac{47}{64}$	0.734375	$\frac{63}{64}$	0.984375
$\frac{1}{4}$	0.250000	$\frac{1}{2}$	0.500000	$\frac{3}{4}$	0.750000	1	1.000000

decimal equivalents. These decimal equivalents are exact numbers; that is, they have not been rounded off. All numbers have been expressed to six decimal places because some of the exact numbers, such as 0.46875, had to be expressed to six places to show the exact value obtained without rounding off. If at any time you don't need six decimal places, round off the value shown in the table. To those numbers which did not have to be expressed to six decimal places (such as 0.75), we have added zeros (0.750000) so that there would be the same number of places throughout the table.

Check Your Learning

56. In multiplication a specific name has been given to the number 1. It is called the _____ _____ for multiplication.

(identity element)

57. Numbers such as 2.45, 5.7, and 23.567 are called _____ decimals.

(mixed)

58. When performing the four basic arithmetic operations use the same rules for decimals as you did for _____.

(integers)

59. The word per cent means _____.

(hundredths)

60. Per cent (is, is not) always a whole number.

(is not)

61. What is the symbol used for the term per cent?

(%)

62. When you use per cent in a calculation you should replace it by an _____ decimal.

(equivalent)

63. To replace a per cent by a decimal, drop the term "per cent" or the symbol % and move the decimal point _____ places to the _____.

(two; left)

64. To change a decimal to per cent multiply the decimal by _____ and then use the term per cent or the symbol % after it.

(100)

65. The decimal .125 (is, is not) equivalent to the fraction $\frac{1}{8}$.

(is)

66. Sometimes whether you use a fraction or its _____ equivalent is simply a matter of personal preference.

(decimal)

Practice Problems

1. Change each of the following to a percent.
   a) .65
   b) .375
   c) .0075
   d) $\frac{61}{100}$
2. Write the following as decimals.
   a) 30%
   b) 0.4%
   c) 2.5%
   d) 13.57%
3. Give the decimal equivalent for each of these fractions. Express your answers to three decimal places.
   a) $\frac{5}{72}$
   b) $\frac{7}{17}$
   c) $\frac{3}{13}$
   d) $\frac{16}{25}$
4. The sales tax in a certain state is 3%. What is the tax on an article that costs $1.15? How much will the customer have to pay in all?
5. In a certain factory, 75% of the 240 workers signed up for a United States savings bond each month. How many workers buy a bond a month?

Powers and Roots of Numbers

Introduction

46. In this section, we define and calculate the powers of numbers. This paves the way for the main feature of this section: the calculation of the square root of a number. Finding the square root of a number may sound like a forbidding process if you have never attempted or mastered it. But it all boils down to a manipulation of the four basic arithmetic operations: addition, subtraction, multiplication, and division. These are operations that you should have already mastered.

Factors

47. You have learned how to multiply any two numbers. The given numbers are known as *factors*, and the result is called the *product*. As you know, a factor may be an integer, a decimal, a mixed decimal, a fraction, or a mixed number. Sometimes you will have to multiply more than two numbers, as in finding the value of the expression $4 \times 5 \times 7.5$. Each number in such an expression is a factor, and the final result is the product of the given numbers. To find the product of more than two factors, first multiply one of the factors by another. Then multiply the result by the third factor; and if there are other factors, continue until all factors have been used. Of course, the factors may be used in any order.

EXAMPLE. Find the product of 72, 56, 16.5, and 0.25.

SOLUTION. If the factors are taken in the order given, the multiplications may be indicated in the following manner:

$$72 \times 56 = 4032$$
$$4032 \times 16.5 = 66528$$
$$66528 \times 0.25 = 16632 \quad \text{Ans.}$$

The Meaning of "Power"

48. Very often you will be required to multiply a number by itself. In other words, you will have to find the product of two factors that are alike. Examples of such work are

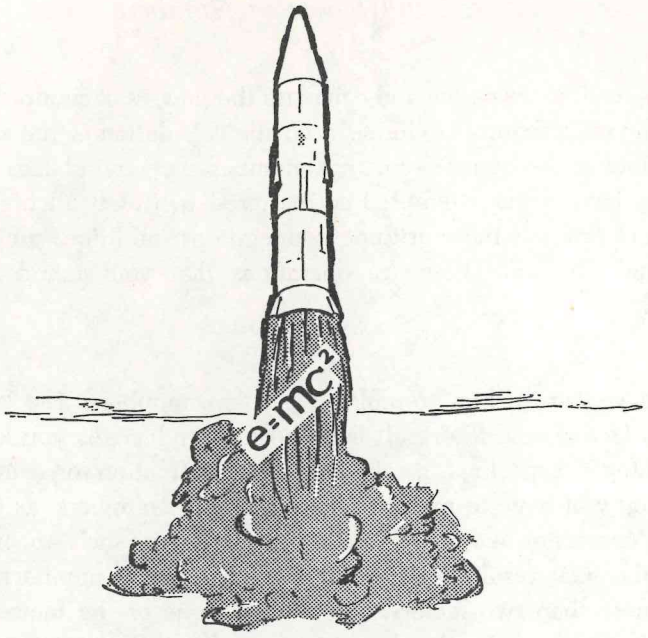
$$4 \times 4 = 16 \qquad 1.2 \times 1.2 = 1.44 \qquad \tfrac{2}{5} \times \tfrac{2}{5} = \tfrac{4}{25}$$

Again, you may have to multiply three factors that are alike, as in the following examples:

$$3 \times 3 \times 3 = 27 \qquad 10 \times 10 \times 10 = 1000 \qquad 0.5 \times 0.5 \times 0.5 = 0.125$$

In certain kinds of work it is necessary to find the product of more than three equal factors. You can find all such products by applying the general method for multiplication.

When all the factors of a product are alike, the product is given a special



name. It is called a *power* of one of the factors. Since $4 \times 4 = 16$, the result 16 is a power of 4; and since $10 \times 10 \times 10 = 1000$, the result 1000 is a power of 10. If there are only two equal factors, the product is the *second power*. Thus 16 is the second power of 4. If there are three equal factors, the product is the *third power*. Thus 1000 is the third power of 10. If there are four equal factors, the product is the *fourth power*, and so on.

When any power of a number is found, the number is said to be *raised* to that power. For example, the work for raising 4 to the second power is $4 \times 4 = 16$; and raising 10 to the third power gives $10 \times 10 \times 10 = 1000$.

EXAMPLE 1. Find the second power of 0.75.

SOLUTION. To raise 0.75 to the second power, multiply 0.75 by 0.75. The work is shown at the right. The second power of 0.75 is 0.5625. Ans.

```
 .75
 .75
 375
 525
.5625
```

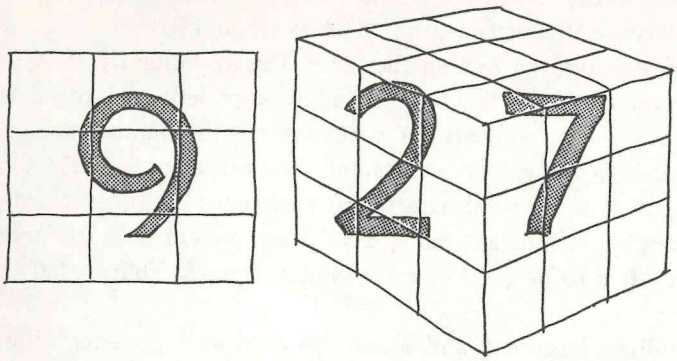
EXAMPLE 2. Raise 2.89 to the third power.

SOLUTION. You must use the given number as a factor three times, as indicated in the expression $2.89 \times 2.89 \times 2.89$. To find the power, multiply 2.89 by 2.89 to get 8.3521. Then multiply 8.3521 by 2.89 to get 24.137569. Ans.

EXAMPLE 3. What is the fourth power of $\frac{1}{2}$?

SOLUTION. Here you must use $\frac{1}{2}$ as a factor four times. The work may be indicated in the following manner:

$$\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{16} \quad \text{Ans.}$$



Practice Problems

1. Find the second power of each of the following numbers:
   a) 3 *b)* 1.5 *c)* 0.8 *d)* 0.02
2. Raise each of the following numbers to the second power:
   a) 32 *b)* $\frac{4}{5}$ *c)* $\frac{8}{3}$ *d)* 40
3. Raise each of the following numbers to the third power:
   a) 7 *b)* 1.2 *c)* 0.6 *d)* 0.04
4. Find the third power of each of the following numbers:
   a) 24 *b)* $\frac{2}{3}$ *c)* $\frac{5}{4}$ *d)* 30

Square and Cube of a Number

49. The second power and the third power of a number have special names. The second power of a number is called the *square* of the number. For instance, since $3 \times 3 = 9$ the product 9 is the square of 3. The third power of a number is commonly known as the *cube* of the number. Thus 27 is the cube of 3.

The process of finding the square of a number, or of multiplying a number by itself, is called *squaring the number*, and we say that the number is squared. The process of finding the cube of a number, or of using a number as a factor three times, is called *cubing the number*, and it is said that the number is cubed.

Exponent

50. Instead of writing the word "square" or the words "second power," it is more convenient to indicate the square, or second power, of a number by writing the number with a small figure 2 to the right of and near the top of the number. For example, the square of 17 is written 17^2 and the square of 8.3 is written 8.3^2. This is a form of mathematical shorthand that tells us how many times a number is to be used as a factor.

Likewise, instead of writing the word "cube" or the words "third power," it is more convenient to indicate the cube or the third power of a number by writing

a small figure 3 to the right of and near the top of the number. For instance, the cube of 9 is written 9^3 and the cube of 0.6 is written 0.6^3.

A figure that is written in a smaller size after and slightly above a number is called an *exponent*. Thus, the 2 in 17^2 is an exponent and the 3 in 0.6^3 is an exponent. The exponent indicates how many times the number ahead of it, called the *base number*, is to be used as a factor. The exponent 2 in 17^2 shows that the base number 17 is to be used as a factor two times. In other words, 17^2 means 17×17. Likewise, 8.3^2 means 8.3×8.3. The exponent 3 in 0.6^3 means that the base number 0.6 is to be used as a factor three times. That is, 0.6^3 means $0.6 \times 0.6 \times 0.6$.

Other numbers besides 2 and 3 may be used as exponents. However, unless you are required to solve certain kinds of problems, you will probably never use these other exponents. But if you should have to use exponents besides 2 and 3, just remember that an exponent shows how many times the base number is to be used as a factor. For example, 2^4 means $2 \times 2 \times 2 \times 2$.

The way to read a power of a number indicated by an exponent is shown by the following examples:

17^2 is read either "seventeen squared" or "seventeen to the second power."

8.3^2 is read either "eight point three squared" or "eight point three to the second power."

9^3 is read either "nine cubed" or "nine to the third power."

0.6^3 is read either "zero point six cubed" or "zero point six to the third power."

2^4 is read "two to the fourth power."

Practice Problems

1. Write the square of each of the following numbers by using an exponent.
   a) 3 *b)* 15 *c)* 0.8 *d)* 0.02
2. Write the cube of each of the following numbers by using an exponent.
   a) 7 *b)* 1.2 *c)* 0 6 *d)* 0.04
3. Write each of the following powers as a product of two or more factors. HINT. $4^2 = 4 \times 4$.
   a) 5^2 *b)* 16.4^2 *c)* 0.7^2 *d)* 11^3
   e) 6.2^3 *f)* 0.03^3 *g)* 6^4 *h)* 8.9^5
4. Write each power in Problem 3 in words. HINT. The power 4^2 is written "four squared" or "four to the second power." Which number is the base? Which is the exponent?

Check Your Learning

67. In multiplication, the given numbers that you are to multiply are called _____.
(factors)

68. When all of the factors of a product are alike the product is called a _____ of one of the factors.

(power)

69. If there are four equal factors the product is to the _____ power.

(fourth)

70. When a power of a number is found the number is said to be _____ to that power.

(raised)

71. The second power of a number is called the _____ of the number.

(square)

72. The cube of 2 is _____.

(2^3 or 8)

73. When you use a number as a factor three times the number is said to be _____.

(cubed)

74. A figure written in a smaller size after and slightly above the number is called a(n) _____.

(exponent)

75. An exponent tells how many times the _____ _____ is to be used as a factor.

(base number)

Powers of Numbers Containing Fractions

51. If the number to be squared contains a fraction and you want to use an exponent, you should indicate the square in this way: First write the number and enclose it in parentheses. Then write the figure 2, in a smaller size, to the right of and near the top of the symbol). For instance, write the square of $\frac{3}{4}$ as $(\frac{3}{4})^2$ and write the square of $7\frac{1}{2}$ as $(7\frac{1}{2})^2$.

You can find the value of $(\frac{3}{4})^2$ as follows

$$(\tfrac{3}{4})^2 = \tfrac{3}{4} \times \tfrac{3}{4} = \tfrac{9}{16}$$

or

$$(\tfrac{3}{4})^2 = \tfrac{3}{4} \times \tfrac{3}{4} = 0.75 \times 0.75 = 0.5625$$

And you can find the value of $(7\frac{1}{2})^2$ as follows

$$(7\tfrac{1}{2})^2 = 7\tfrac{1}{2} \times 7\tfrac{1}{2} = \tfrac{15}{2} \times \tfrac{15}{2} = \tfrac{225}{4}$$

or

$$(7\tfrac{1}{2})^2 = 7\tfrac{1}{2} \times 7\tfrac{1}{2} = 7.5 \times 7.5 = 56.25$$

Note the need for the parentheses where there is a fraction in the number to be raised to a power. Suppose you want to write the square of $\frac{3}{4}$ and you omit the parentheses. The result will be $\frac{3^2}{4}$. Some other person seeing your work would, of course, assume that only the 3 is to be squared and that the square of 3 is to be divided by 4. It would, therefore, be reasonable for this person to carry out the following steps:

$$\frac{3^2}{4} = \frac{3 \times 3}{4} = \frac{9}{4}$$

But $\frac{9}{4}$ is not the square of $\frac{3}{4}$. As shown in the preceding paragraph, the correct value of the square of $\frac{3}{4}$ is $\frac{9}{16}$.

If you want to write the square of $7\frac{3}{4}$ and you omit the parentheses, the result is $7\frac{3^2}{4}$. This is wrong. The correct way to write the square of $7\frac{3}{4}$ is $(7\frac{3}{4})^2$.

You should indicate any other power of a number containing a fraction in the manner just described for a square. Enclose the number in parentheses, and write the proper exponent after and near the top of the symbol). For example, write the cube of $\frac{5}{8}$ as $(\frac{5}{8})^3$ and write the cube of $11\frac{3}{4}$ as $(11\frac{3}{4})^3$.

You can see from the foregoing explanations that $(\frac{5}{8})^3$ means $\frac{5}{8} \times \frac{5}{8} \times \frac{5}{8}$ and that $(1\frac{1}{2})^4$ means $1\frac{1}{2} \times 1\frac{1}{2} \times 1\frac{1}{2} \times 1\frac{1}{2}$.

Perfect Squares

52. A number that is the product of two equal factors is a *perfect square*. For example, the numbers 1, 25, 81, $2\frac{1}{4}$, $\frac{4}{16}$, 3.61, and 0.0625 are perfect squares. Each of these numbers is the product of two equal factors, as follows:

Two Equal Factors		Perfect Squares
1×1	=	1
5×5	=	25
9×9	=	81
$\frac{3}{2} \times \frac{3}{2}$	=	$\frac{9}{4}$ or $2\frac{1}{4}$
$\frac{2}{4} \times \frac{2}{4}$	=	$\frac{4}{16}$
1.9×1.9	=	3.61
0.25×0.25	=	0.0625

You can see that a perfect square may be a whole number, a decimal, a mixed decimal, a fraction, or a mixed number.

TABLE 2

SQUARES OF SIMPLE NUMBERS

Number	Square	Number	Square	Number	Square
1	1	8	64	15	225
2	4	9	81	16	256
3	9	10	100	17	289
4	16	11	121	18	324
5	25	12	144	19	361
6	36	13	169	20	400
7	49	14	196		

You should realize, from the definition at the beginning of this article, that a perfect square has two equal factors. For example, the perfect square 25 has two equal factors, 5 and 5.

Perfect Squares of Some Whole Numbers

53. You should be able to recall quickly the square of any whole number from 1 to 20. It is often helpful to know these squares. For your convenience they are listed in Table 2. The numbers from 1 to 20 are in the columns headed Number, and the square of each of these numbers is in the column to the right headed Square. Thus Table 2 shows that the square of 4 is 16, the square of 7 is 49, and the square of 9 is 81. Knowing these numbers and their squares will be of great benefit to you in working with square roots.

Practice Problems

1. Find the value of
   a) $(\frac{1}{7})^3$
   b) $(\frac{2}{3})^4$
   c) $(\frac{4}{5})^5$
   d) $(\frac{7}{11})^3$
2. What is the square of these numbers?
   a) .7
   b) .11
   c) .21
   d) .018

Use of Powers of 10

54. You can readily see that $10^2 = 100$, that $10^3 = 1000$, that $10^4 = 10{,}000$, and so on. Notice that in each power of 10, the number of zeros following the 1 is equal to the exponent of 10. It follows that any whole number ending with two or more zeros can be considered as the product of the digits to the left of the zeros and the power of 10 indicated by the zeros. For example, 3200 can be considered as 32×100 or 32×10^2, and 4,000,000 can be considered as $4 \times 1{,}000{,}000$ or 4×10^6. This method of writing a number is often used in some kinds of work. It is called writing a number as a power of 10.

When there is more than one digit to the left of the zeros in a given number, it is sometimes preferable to write the digits of the number preceding the power of 10 to form a mixed decimal between 1 and 10 and to use a power of 10 greater than the given number of zeros. For example, 3200 may be treated as 3.2×1000 and written 3.2×10^3, instead of 32×10^2. Likewise, 3,140,000 would often be considered as $3.14 \times 1{,}000{,}000$ and written 3.14×10^6.

Check Your Learning

76. When writing the square of a fraction, you must enclose the fraction within a _____ and use the exponent 2.

(parentheses)

77. The correct way to write the cube of $10\frac{1}{4}$ is _____.

$(10\frac{1}{4})^3$

78. A number that is the product of two equal factors is a _____ _____.

(perfect square)

79. A perfect square has _____ equal factors.

(two)

80. In each power of 10, the number of zeros following the 1 is equal to the exponent of _____.

(10)

81. To write 4000 using powers of 10 you have $4 \times$ _____.

(10^3)

What a Square Root Is

55. To solve a problem, you may have to find the square root of a number. The square root of any number that is a perfect square is one of the two equal factors of the number. A few simple examples follow:

Since $3 \times 3 = 9$, the square root of 9 is 3.

Since $8 \times 8 = 64$, the square root of 64 is 8.

Since $2.4 \times 2.4 = 5.76$, the square root of 5.76 is 2.4.

Since $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, the square root of $\frac{4}{9}$ is $\frac{2}{3}$.

We urge you to memorize the squares of the numbers 1 to 20 in Table 2 so that you can immediately recognize the square of any one of these numbers. There is a second way in which you can use this table of perfect squares. Once you learn that $18^2 = 324$, you also know that the square root of 324 is 18. Thus if you need the square root of a perfect square, you will already have it.

In the following articles we shall show you how to find the square root of a number. When a number is not a perfect square, the square root cannot be determined exactly. It can, however, be found to any desired number of places.

The process of finding the square root of a number is sometimes called *taking the square root* or *extracting the square root.*

The Radical Sign

56. The usual way to indicate that the square root of a number is to be found is to place the symbol $\sqrt{\ }$ in front of the number and to draw a line over the number. For example, $\sqrt{64}$ means the square root of 64 and $\sqrt{5.76}$ means the square root of 5.76. Thus $\sqrt{64} = 8$ and $\sqrt{5.76} = 2.4$. The symbol $\sqrt{\ }$ is called the *radical sign,* and the line used with it is the bar, or vinculum, which has already been referred to in this text. The number written under the bar is called the *radicand.* For example, in $\sqrt{64}$, 64 is the radicand. Sometimes, the radical sign and the bar $\sqrt{\ \ }$ together are called a radical sign.

Other Roots

57. As we have said, when a number is used as a factor three times, the power is called the cube of the given number. One of the three equal factors

of that power is called the cube root of the power. For example, since $4^3 = 4 \times 4 \times 4 = 64$, you can see that the cube root of 64 is 4. Similarly, since $1.2^3 = 1.2 \times 1.2 \times 1.2 = 1.728$, the cube root of 1.728 is 1.2. As a final example $(\frac{2}{3})^3 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$, the cube root of $\frac{8}{27}$ is $\frac{2}{3}$.

The usual way to indicate that the cube root of a number is to be found is to use the radical sign and a bar, as for square root, and to place a small digit 3 in the opening in the radical sign. Thus $\sqrt[3]{64}$ means the cube root of 64, and $\sqrt[3]{1.728}$ means the cube root of 1.728. The 3 is called the *index*. For square root the index is 2 but it is not written: Thus $\sqrt{81} = 9$.

It is also possible to have a fourth root of a number, a fifth root, or any other root. The fourth root of a number is one of the four equal factors from which the given number would be obtained by multiplication. For instance, the fourth root of 16 is 2, since $2^4 = 2 \times 2 \times 2 \times 2 = 16$. The fifth root of a number is one of its five equal factors. For example, since $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$, the fifth root of 32 is 2. Likewise, the sixth root of a number is one of its six equal factors, and so on.

The fourth root, fifth root, or other root of a number can be indicated by using a radical sign, a bar, and the proper index in the opening of the radical sign. Thus $\sqrt[4]{16}$ means the fourth root of 16 and $\sqrt[5]{32}$ means the fifth root of 32.

Although we have mentioned other roots, we will discuss only the ways of finding the square root of a number. There are several ways in which the square root of a number may be found. In this text, we will limit our discussion to two methods of finding square root: the table method and the calculation method. You will see that either method has its limitations.

Check Your Learning

82. The process of finding the square root of a number is sometimes called _____ the square root.

(taking; extracting)

83. Write the symbol used to indicate that the square root is to be found.

($\sqrt{\ }$)

84. The number written under the bar is called the _____.

(radicand)

85. The cube root of 27 can be shown as $\sqrt[3]{27}$. The 3 is called the _____.

(index)

Finding Square Root by Table Method

58. You will often be able to save time and work in finding the square root of some numbers by using a table such as Table 3, at the back of this text. We will now show you how to use the table for finding the square root of any whole number from 1 to 1000. The columns headed Number in the table contain all the

whole numbers from 1 to 1000, and the columns headed Square Root contain the corresponding square roots. Each square root in the table has four digits. In the square root of any number from 1 to 99 there is only one digit to the left of the decimal point. So the square root of each number less than 100 in the table is given to three decimal places. In the square root of any number from 100 to 1000 there must be two digits to the left of the decimal point. Therefore, the square root of 100 or a larger number in the table is given to two decimal places. These values are accurate enough for most of the problems that you will meet in practical work.

You can get the square root of a whole number from 1 to 1000 simply by looking at Table 3. First locate the given number in one of the columns headed Number, and then take the square root from the next column to the right, headed Square Root. For example, suppose that you want to find the square root of 724. Locate 724 in a column headed Number. Then, in the next column to the right headed Square Root and opposite 724, you will find 26.91. This is the value of the square root of 724 to two decimal places. In like manner the square root of 125 is found to be 11.18, and the square root of 378 is found to be 19.44.

As you probably realize, the usefulness of Table 3 for finding the square roots of numbers is limited. You cannot use it to find the square roots of mixed numbers or mixed decimals between 1 and 1000. For example, you can't find the square root of 739.4 or $439\frac{3}{8}$ directly from the table. Again, you can't use the table directly to find the square roots of numbers greater than 1000. For instance, you can't find the square root of 1729 from the table.

Practice Problems

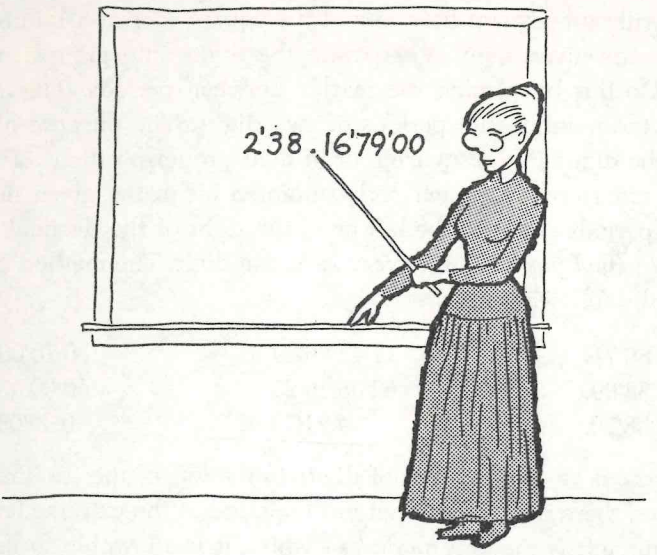
Find the square root of these numbers from the table in the back of this text.

1. 368
2. 787
3. 556
4. 635
5. 874

Calculation Method

59. You have seen some of the disadvantages of the table method of finding square root. The method we are now going to present, the calculation method, overcomes the disadvantages of the table method. You can find the square root of any number by the calculation method. As we stated in a preceding article, an exception to this is a number that is not a perfect square. But even for such a number, the square root can be found to any desired number of places.

Thus you can see the wide application of the calculation method in extracting (finding) the square roots of numbers. That is why we discuss this method so thoroughly in the articles that follow and why you should thoroughly master the method.



Writing Number for Finding Square Root

60. The steps in finding the square root of a number by the calculation method are the same whether the given number is a whole number, a mixed decimal, or a decimal. Remember that in these texts of the *Practical Arithmetic* series we use the term "decimal" by itself to mean a number having a value less than 1. A number greater than 1 with a decimal part is called a *mixed decimal,* and a number greater than 1 with a fractional part is called a *mixed number.* When the given number is a mixed number, the fractional part should be changed to a decimal before the work of finding the square root is started.

When the given number is a fraction, the square root can always be found by changing the fraction to a decimal and getting the square root of the decimal. However, if the denominator of the fraction is a perfect square, some people find the square root of the numerator and the square root of the denominator, and then divide the square root of the numerator by the square root of the denominator.

To find the square root of a number by the calculation method, first write the given number and draw a straight line over the number, just as if the number were the dividend in long division. You can write the digits in the square root above the line one at a time, just as you write the digits in the quotient in long division. When writing the given number, show the decimal point in the proper place. But do not include any commas when there are four or more digits in front of the decimal point. Even if the given number is a whole number, it is not likely to be a perfect square. Therefore, the square root will usually have a decimal part. So you should put a decimal point in a whole number after the digit in the units place.

Pointing Off

61. We will now explain how to find the square root of a number. After you have written the given number, separate the digits into groups, or periods, of two digits. Do this by placing the mark ′ between periods. This step is called *pointing off* the number into periods of two digits. The purpose of pointing off is to locate the digits in the square root in their proper positions. There will be a digit in the square root for each period pointed off in the given number. When pointing off periods either to the left or to the right of the decimal point, *always start at the decimal point.* Count every zero as a digit. The method of pointing off is shown in the following numbers:

86′71.	1′42.76′39	0.70′06
6′33′09.	62′35.40′25	0.00′43
42′06′38′70.	4.21′73′02	0.00′06′25

When there is an odd number of digits to the left of the decimal point in the given number, there is only one digit in the period at the extreme left. The period at the extreme left is the only period in which it is allowable to have one digit. Thus, when the number 63309 is pointed off, the digit 6 is treated as a period by itself. Likewise, the digit 1 in 142.7639 is treated as a period, and the digit 4 in 4.217302 is treated as a period.

If there is an odd number of digits, including zeros, after the decimal point in the given number, you must write a zero at the end of the decimal part to make a complete period of two digits. For example, you should point off 1.196 as 1.19′60′, you should point off 0.03745 as 0.03′74′50, and you should point off 0.00228 as 0.00′22′80.

For the purpose of pointing off, the decimal point is considered a separation mark; so you need not insert the mark ′ above the decimal point.

Check Your Learning

86. The calculation method for finding the square root of a number is the _____ whether the given number is a whole number, a mixed decimal, or a decimal.

(same)

87. To find the square root of a fraction, change the fraction to a decimal and then find the _____ _____ of the decimal.

(square root)

88. To find the square root of a number, first separate the number into groups of _____ digits.

(two)

89. There will be a digit in the square root for each _____ that is pointed off in the given number.

(period)

90. When pointing off periods either to the left or to the right of the decimal point, always start at the _____ _____.

(decimal point)

91. When a number is pointed off, every zero (is, is not) counted as a digit.

(is)

Basic Steps in Finding Digits in Square Root

62. When you have pointed off the given number into periods of two digits, locate the decimal point in the square root above the decimal point in the given number. You are then ready to carry out the steps for finding the various digits in the square root. To explain these steps to you, we will solve several examples. In many ways the work is similar to that of long division.

EXAMPLE 1. Find the square root of 31,505,769.

SOLUTION. First write the given number and point it off. Then draw the line above it, and put the decimal point in the square root directly above the decimal point in the given number, like this:

$$\overline{.}$$
$$31'50'57'69.$$

To find the digits in the square root, you must start with the period of digits at the extreme left in the given number.

The first digit in the square root is the largest number whose square is not greater than the figure in the first period. The number in the first period in this example is 31, and the largest square smaller than 31 is 25. Hence the first digit in the square root is 5, which is the square root of 25.

$$\begin{array}{r} 5 \\ 5\overline{)31'50'57'69.} \\ \underline{25} \\ 6 \end{array}$$

Write the first digit in the square root above the first period in the given number, as shown. Then write the same digit to the left of the given number, as if it were a divisor in division. Next write the square of the first digit under the first period and subtract. As shown above, 25 is written under 31 and is subtracted from 31, and the difference 6 is written below the 25.

Now find the second digit in the square root this way: Bring down the two digits in the second period of the given number and write them after the 6 obtained by subtraction in the preceding step. Thus, the digits 5 and 0 of the second period are written after the 6. The next step is to multiply the first digit in the square root by 20, and write the product as a second trial divisor. Thus, write the product of 5 and 20, which is 100, to the left of 650, as shown here:

$$\begin{array}{r} 5 \\ 5\overline{)31'50'57'69.} \\ \underline{25} \\ 100\overline{)\ 6\ 50} \end{array}$$

When you find the second digit of the square root in this example, you will add it to the 100. To find the second digit, divide the 650 by 100, and use the whole number

```
    5  6      .
5/31'50'57'69.
  25
106/ 6 50
     6 36
     ----
       14
```

thus obtained as a trial value. This number is 6. As shown in the work, write a 6 as the second digit in the square root above the second period of the given number, and also add a 6 to the 100 in the divisor to make the complete divisor 106. Now multiply this complete divisor by 6 and write the product 636 below 650. Subtract 636 from 650 to get 14.

Here the product 636 is less than the 650, so 6 is the correct second digit in the square root. We will show you in the next example what to do when the product is too large.

The steps for finding the third digit in the square root are similar to those described for finding the second digit. As shown, bring down the digits 5 and 7 in the third period and write them after the 14. Also, multiply the known part (56) of the square root by 20 to get 1120 and write this product as a part of the third divisor. To the 1120 you will add the third digit of the square root, when it is found. To find the third digit, divide 1457 by the 1120. The whole number thus obtained is 1. Write a 1 as the third digit in the square root above the third period of the given number, and also add a 1 to the 1120 in the divisor. Then multiply the complete divisor 1121 by the third digit 1 in the square root, and write the product 1121 below the 1457. Subtract it to get 336. Your work will look like this:

```
    5  6  1    .
5/31'50'57'69.
  25
106/ 6 50
     6 36
     ------
1121/  14 57
       11 21
       -----
        3 36
```

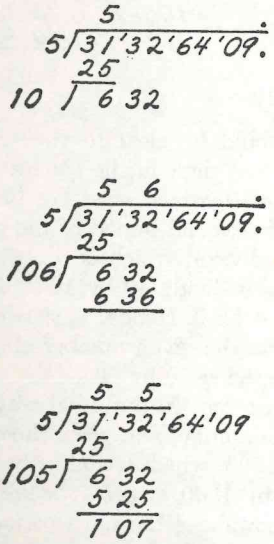
The fourth digit in the square root is found in a similar way. Bring down the digits 6 and 9 in the fourth period of the given number and write them after the 336. Also, multiply the known part 561 of the square root by 20 to get 11220, and write this product as a part of the fourth divisor. To find the fourth digit of the square root, divide 33669 by the 11220. Write the digit 3, thus found, in the square root above the fourth period of the given number and also add it to the 11220 in the divisor. Then multiply the complete divisor 11223 by 3 and write the product 33669 below 33669 and subtract it. Since the difference is zero and there are no other periods to bring down, the solution is completed. Hence the square root of 31,505,769 is 5613. Here is the complete solution:

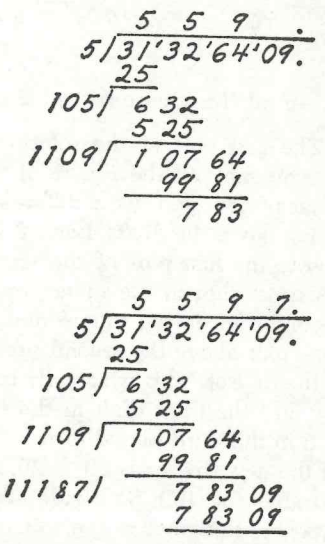
```
    5  6  1  3.
5/31'50'57'69.
  25
106/ 6 50
     6 36
     ------
1121/  14 57
       11 21
       -----
11223/  3 36 69
        3 36 69
        -------
```

You can check the result by multiplying 5613 by 5613 to see if the product is 31,505,769.

EXAMPLE 2. Find the square root of 31,326,409.

SOLUTION. As shown in the work below, the first part of the solution is similar to that in Example 1. Write the given number and point it off. Find the first digit in the square root, which is 5. Subtract the square of 5 from 31 to get 6. Bring down the digits in the second period of the given number and write them after the 6. Multiply the first digit 5 in the square root by 20 to get 100 as the first part of the second divisor.



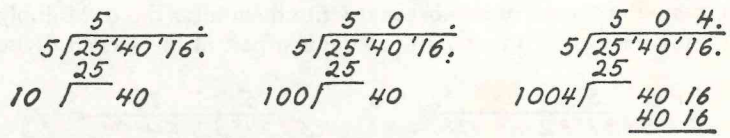


When you divide 632 by 100 to find the second digit in the square root, the whole-number part of the quotient is 6. So you may write 6 in the square root and in the divisor, as shown. However, when you multiply the completed divisor 106 by 6, the product is 636. Since this product is greater than the 632, the second digit in the square root cannot be 6, but must be smaller. You should therefore erase the 6 and substitute 5 in the square root and in the divisor. Then write the product of 105 and 5, or 525, below the 632 and subtract 525 from 632. It so happens that the difference 107 obtained by subtracting 525 from 632 is greater than the divisor 105. In long division, when the difference is found to be greater than the divisor, it is necessary to increase the last digit in the quotient. In finding a square root, however, it is possible to have a difference slightly greater than the divisor and still have the correct last digit in the square root. To avoid writing an incorrect digit in the square root and the divisor when there is doubt about the correct digit, make the trial calculation on scratch paper before you insert the digit in the solution.

The solution is continued as shown. Bring down the digits 6 and 4 in the third period of the given number, and write them after the 107. Then multiply the known part 55 of the square root by 20 to get 1100 as a part of the third divisor.

Next, divide 10764 by 1100 to get 9 as the third digit in the square root. Write this digit in the square root above the third period in the given number, and add it to the 1100 in the divisor. Now write the product of 1109 and 9, or 9981, below the 10764 and subtract it to get 783.

The complete solution is shown. You find the fourth digit by carrying out the usual steps. Write the digits 0 and 9 in the fourth period of the given number after the 783; multiply the known part 559 of the square root by 20 to get 11180, which is a part of the new divisor. Divide 78309 by 11180 to get 7, which is tried as the fourth digit of the square root and the last digit of the divisor. Since $11187 \times 7 = 78309$, the solution is completed. The square root of 31,326,409 is 5597. Ans.



EXAMPLE 3. Find the square root of 254,016.

SOLUTION. The first part of the solution should be clear to you without an explanation. When you subtract the square of the first digit in the square root from the number in the first period, you get a difference of 0, which need not be written. The next step is to bring down the digits 4 and 0 in the second period of the given number, as shown, and write the first part of the second divisor, which is 5×20, or 100.

To find the second digit in the square root, you divide 40 by 100. Since 40 is less than 100, the second digit in the square root must be 0. Hence, as shown, you write a zero in the square root above the second period in the given number and place a zero after the 100 in the divisor. This is actually multiplying 50 by 20.

You can now find the third digit in the square root in the usual way. Bring down the digits 1 and 6 in the third period of the given number and write them after the 40. The first part of the new divisor is 50×20, or 1000, which has already been obtained by putting the 0 after the 100. So divide 4016 by 1000 to get 4 as the third digit in the square root. Write this digit in the square root above the third period in the given number and add it to 1000 in the divisor. When you multiply the complete divisor 1004 by the 4, the product is 4016. The solution is completed. The square root of 254,016 is 504. Ans.

You can check the result by multiplying 504 by 504 to see if the product is 254,016.

Check Your Learning

92. To find the digits in the square root, start with the period of digits at the extreme _____ in the given number

(left)

93. The first digit in the square root is the largest number whose square is not _____ than the figure in the first period.

(greater)

94. For any trial division multiply the digit or digits already found in the root by _____.

(20)

95. A _____ divisor is formed by adding the last digit found in the root to the last trial divisor.

(complete)

Practice Problems

Find the square root of each of the following numbers:

1. 1369	2. 196	3. 3249	4. 43,264
5. 136,161	6. 30,276	7. 1,155,625	8. 19,749,136

Square Root of Decimal or Mixed Decimal

63. In each of the following examples there are digits to the right of the decimal point in the given number and in the square root. But finding each digit in the square root is the same as when the given number is a whole number.

EXAMPLE 1. Find the square root of 677.5609.

```
      2  6. 0  3
   2 ) 6'77.56'09
       4
  46 ) 2 77
       2 76
 5203      1 56 09
           1 56 09
```

SOLUTION. Before you try to find the digits in the square root, point off the given number and locate the decimal point in the square root. Then find the digits in the square root to the left of the decimal point by using the method explained previously. These digits are 2 and 6.

To find the digits to the right of the decimal point in the square root, proceed as if the given number were 6,775,609. In finding the digits in the square root, the decimal point between two periods has the same effect as the mark ' in a whole number. No matter where the decimal point is located in the square root, use every divisor as if it were a whole number. As shown, the square root of 677.5609 is 26.03. Ans.

You can check the result by multiplying 26.03 by 26.03. In this example the given number is a perfect square.

EXAMPLE 2. Find the square root of 0.5625.

SOLUTION. After you have pointed off the given number and have located the decimal point in the square root properly, find the digits in the square root as if the given number were 5625. Although the decimal point in the square root comes before the digit 7, do not put a decimal point in either divisor. As shown, the square root of 0.5625 is 0.75. Ans.

When finding the square root of a decimal, be careful to point off the periods and to locate the decimal point correctly. Also, remember that the square root of a decimal is larger than the number itself. For instance, the square root of 0.5625 is larger than 0.5625. The reason becomes evident when you stop to think that the square of 0.75 or of any other decimal is found by multiplying that decimal by a number less than 1. Thus the square of a decimal must be less than the given decimal, and the square root of any decimal must be larger than the given decimal.

```
      . 7  5
   7 ) .56'25
       49
 145 )  7 25
        7 25
```

EXAMPLE 3. Find the square root of 0.000576.

SOLUTION. In this example the first period after the decimal point consists of two zeros, so the first digit to the right of the decimal point must be zero. You can find the other digits in the square root as if the given number were 576. Although the second period after the decimal point consists of the digits 0 and 5, you need not use the zero when finding the digit 2 in the square root. The square root of 0.000576 is 0.024. Ans.

As a check, $0.024 \times 0.024 = 0.000576$.

```
    .0 2 4
  2/.00'05'76
      4
  44/  1 76
       1 76
```

Square Root of Nonperfect Squares

64. In each of the preceding examples, the given numbers are perfect squares. When the given number is not a perfect square, the methods of locating the decimal point and finding the digits in the square root are the same as the ones described. However, you may have to write zeros to the right of the decimal point in the given number to obtain the desired number of decimal places in the square root.

When you must find digits to the right of the decimal point in the square root of a number, you will know in advance how many decimal places are needed. You should carry out the calculations to one extra place and then round off the result, as in division. If the extra digit is less than 5, drop it without changing the digit ahead of it. If the extra digit is 5 or greater than 5, drop it but increase the digit ahead of it by 1. You should always check the square root by multiplying the answer by itself. We will work a few examples.

EXAMPLE 1. Find the square root of 3 to three decimal places.

SOLUTION. Since you want to find the digit in the fourth decimal place, you must write four periods of zeros after the decimal point in the given number, as shown in the work. After you have pointed off the number and have located the decimal point in the square root, find the digits in the square root in the usual manner. The digit in the fourth decimal place in the square root is 0; so this digit is dropped. The square root of 3 to three decimal places is 1.732. As a check, $1.732 \times 1.732 = 2.999824$.

```
     1. 7 3 2 0
  1/3.00'00'00'00
     1
  27/2 00
     1 89
 343/  11 00
       10 29
3462/     71 00
          69 24
34640/     1 76 00
```

EXAMPLE 2. Find the square root of 0.3 to four decimal places.

SOLUTION. In this example you must complete the first period to the right of the decimal point in the given number by writing a zero after the 3. Then write enough other zeros to make five periods, as shown. Since the digit in the fifth decimal place in the square root is 2, which is less than 5, it is dropped. So the square root of 0.3 to four decimal places is 0.5477. Ans.

As a check, $0.5477 \times 0.5477 = 0.29997529$.

```
        .5  4  7  7  2
    5 ).30'00'00'00'00
        25
 104)    5 00
         4 16
1087)      84 00
           76 09
10947)      7 91 00
            7 66 29
109542)       24 71 00
              21 90 84
```

If you had wanted to find the square root of 0.3 to only three decimal places, you would have stopped the work when you found the fourth digit. Since this fourth digit is 7, which is greater than 5, you would have rounded off the square root by dropping the 7 in the fourth decimal place and changing the 7 in the third decimal place to 8. The result would then have been 0.548.

Let us summarize all that we have said about finding the square root of any number. The procedure is always the same. We will illustrate it by finding the square root of 204.547204.

```
                                  2'04.54'72'04 | 14.302   Ans.
                                  1
                                  104                     first dividend
First complete divisor      24    96
                                    854                   second dividend
Second complete divisor     283   849
                                    572                   third dividend
Third complete divisor      2860  000
                                    57204                 fourth dividend
Fourth complete divisor     28602 57204
```

1. Separate the number into periods of two digits each. Begin at the decimal point and count toward the left for the whole-number part and toward the right for the decimal part.

2. Find the number whose square is equal to or just less than the first period. Write it as the first digit of the root.

3. Subtract the square of the number found in step 2 from the first period and bring down the next period to form a new dividend.

4. Find the trial divisor by multiplying the root already found by 20.

5. Mentally divide the dividend by the trial divisor and use this figure as the next digit of the root. If the trial divisor is greater than the dividend, zero will be the next digit of the root.

6. Add the last digit of the root to the trial divisor. This will give you the complete divisor.

7. Multiply the complete divisor by the last digit of the root and subtract the product from the dividend. If this product is larger than the dividend, decrease the digit found in step 5 by 1 and do steps 6 and 7 again.

8. Bring down the next period and write it after the remainder you found in step 7. Then repeat steps 4 to 8 until you have all the required digits of the root.

Check Your Learning

96. The square root of a decimal is (smaller, larger) than the number itself.
(larger)

97. To find the square root of a number that isn't a perfect square, carry your results to _____ extra place than is required in your answer and then round off.
(one)

98. You should check your answer in square root by _____ the answer by itself.
(multiplying)

Practice Problems

Find the square root of each of the following numbers. Where the number is not a perfect square, round off the square root to three decimal places.

1. 1866.24	2. 0.011696	3. 198.148	4. 205.0624
5. 2.375	6. 4.570428	7. 108	8. 0.006

Square Roots of Fractions

65. As stated previously, you can find the square root of a fraction in this way: Change the fraction to a decimal and then find the square root of the decimal. For example, to find the square root of $\frac{9}{16}$, change the fraction to 0.5625. Then, as shown in Example 2, Art. 63, this square root is 0.75.

You can also compute the square root of any fraction in this way: First find the square root of the numerator and the square root of the denominator and then divide the numerator by the denominator. For example, we'll find the square root of $\frac{9}{16}$. Since $\sqrt{9} = 3$ and $\sqrt{16} = 4$, the square root of $\frac{9}{16}$ is $\frac{3}{4}$. We then divide 3 by 4 to obtain 0.75, the square root of $\frac{9}{16}$.

Finding the square root of a fraction by dividing the square root of the numerator by the square root of the denominator is most useful when the denominator of the fraction is a perfect square. Sometimes if the denominator is not

a perfect square, it is a simple matter to rewrite the fraction with a denominator that is a perfect square. Since there is only one step involved, it is usually worthwhile to make this change, because it saves time and simplifies the calculations. Examples 3 to 5 will show you how to apply this method.

EXAMPLE 1. Find the square root of $\frac{17}{36}$, rounding off the result to three decimal places.

SOLUTION. You should see at a glance that the denominator 36 is a perfect square and its square root is 6, so you should find the square root of the numerator and divide that square root by 6. Since the square root of 17 carried to four decimal places (one more than the number of places required in the answer – Art. 64) is 4.1231, the solution may be indicated as follows:

$$\sqrt{\frac{17}{36}} = \frac{\sqrt{17}}{\sqrt{36}} = \frac{4.1231}{6} = 0.6871$$

When the square root is rounded off to three decimal places, the result is 0.687. Ans.

EXAMPLE 2. Find the square root of $\frac{725}{64}$ to three decimal places.

SOLUTION. Since 64 is a perfect square, you should find the square root of the numerator and divide it by 8, which is the square root of 64. The square root of 725 to four decimal places is 26.9258. The solution may be indicated in this way:

$$\sqrt{\frac{725}{64}} = \frac{\sqrt{725}}{\sqrt{64}} = \frac{26.9258}{8} = 3.3657$$

This result is rounded off to 3.366. Ans.

EXAMPLE 3. What is $\sqrt{\frac{5}{8}}$ to two decimal places?

SOLUTION. The denominator, 8, is not a perfect square. However, we can very easily make it a perfect square by multiplying it by 2. When we do, we must, of course, multiply the numerator by 2. This is how the work would be done:

$$\sqrt{\frac{5}{8}} = \sqrt{\frac{10}{16}} = \frac{\sqrt{10}}{4} = \frac{3.162}{4} = 0.79$$

EXAMPLE 4. What is the square root of $\frac{7}{11}$ to three decimal places?

SOLUTION. Here we multiply numerator and denominator by 11 to make the denominator a perfect square.

$$\sqrt{\frac{7}{11}} = \frac{\sqrt{77}}{11} = \frac{8.7750}{11} = 0.7977$$

Expressed to three decimal places, $\sqrt{\frac{7}{11}} = 0.798$ Ans.

EXAMPLE 5. Calculate the square root of $\frac{33}{45}$ to two decimal places.

SOLUTION. In this example instead of looking for a multiplier to make 45 a perfect square, you should recognize that $45 \div 5 = 9$, which is a perfect square. This time we use division to make the denominator a perfect square. Therefore, the work will be as follows:

$$\sqrt{\frac{33}{45}} = \sqrt{\frac{6.6}{9}} = \frac{2.569}{3} = 0.856 \text{ or } 0.86 \quad \text{Ans.}$$

Check Your Learning

99. When the denominator of a fraction is a _____ _____ the simplest way to find the square root of the fraction is to divide the square root of the numerator by the square root of the denominator.

(perfect square)

100. Sometimes if the denominator is not a perfect square it (is, is not) a simple matter to rewrite the fraction with a denominator that is a perfect square.

(is)

101. To rewrite the fraction $\frac{2}{3}$ so that the denominator is 9, a perfect square, you must multiply the _____ of the fraction by _____.

(terms; 3)

Practice Problems

Find the square root of each of the following fractions. Round off the result to three decimal places if the fraction is not a perfect square.

1. $\frac{49}{25}$
2. $\frac{13}{4}$
3. $\frac{145}{16}$
4. $\frac{5}{6}$
5. $\frac{3}{7}$
6. $\frac{5}{32}$

IMPORTANT CHAPTER.

Applications of Squares and Square Roots

Finding Surface Areas

66. The principles explained in the preceding sections have many practical applications. However, these methods or techniques must be mastered before you try to apply them. Assuming that you have acquired this facility, we will now consider some problems which involve squares and square roots.

Squares of numbers are used in finding the areas of flat surfaces such as squares and circles. A square is a figure that has four equal sides and angles, each angle being a right angle, as shown in Fig. 1. The area of a square is the square of the length of one of its sides. If this length is stated in inches, the area is expressed in square inches; if in feet, the area is expressed in square feet; and so on.

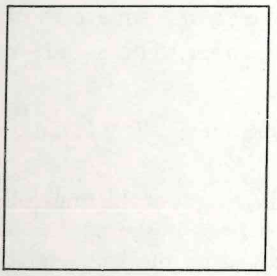


FIG. 1

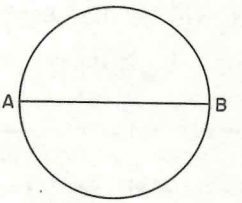


FIG. 2

A circle is a figure bounded by a curved line all points of which are at the same distance from the center, as shown in Fig. 2. The curved boundary line is called the *circumference*. A straight line drawn through the center, with its ends A and B on the circumference, is called the *diameter* of the circle. The area of a circle may be found by multiplying the square of its diameter by 0.7854.

EXAMPLE 1. The surface of a skating rink is in the form of a square that measures 124 ft (feet) on a side. What is the area of the surface?

SOLUTION. The area of a square is equal to the square of one side, so the area of the surface is 124^2, or $124 \times 124 = 15{,}376$ sq ft (square feet). Ans.

EXAMPLE 2. A circular plate of glass is 15 in. (inches) in diameter. What is its area? Give answer to the nearest tenth of a square inch.

SOLUTION. The area of a circle is 0.7854 times the square of the diameter. Therefore, the area of the circular plate is $0.7854 \times 15^2 = 0.7854 \times 225 = 176.715$ or 176.7 sq in. Ans.

Right Triangles

67. A right triangle is a three-sided figure in which two of the sides meet at a right angle. Such a triangle is shown in Fig. 3. The sides AC and BC form a right angle at C. The longest side, AB, of the right triangle is called the hypotenuse. In a right triangle the sum of the squares of the lengths of the two short sides is

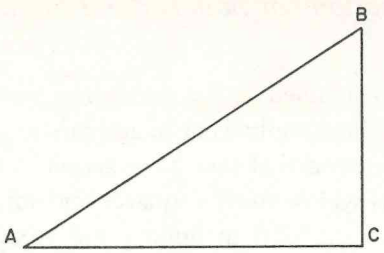


FIG. 3

equal to the square of the length of the longest side, or hypotenuse. This important property of the right triangle enables us to find any one of the sides of the triangle if the other two sides are given. For example, if one short side and the longest side are known and the third side is to be found, square the two known sides and subtract the smaller square from the larger. The square root of the difference will be the length of the third side.

EXAMPLE 1. If the side *AB*, Fig. 3, is 10 ft long and *BC* is 6 ft long, what is the length of *AC*?

SOLUTION. It has just been stated that to find the length of the third side of a right triangle, you square the longest side and subtract from it the square of the other given side. Then the square root of this difference will be the length of the required side.

$$\sqrt{10^2 - 6^2} = \sqrt{100 - 36} = \sqrt{64} = 8 \text{ or } 8 \text{ ft} \quad \text{Ans.}$$

EXAMPLE 2. A wall standing on level ground is braced by a slanting timber whose lower end is 15 ft from the base of the wall and whose upper end is 20 ft above the ground. What is the length of the brace?

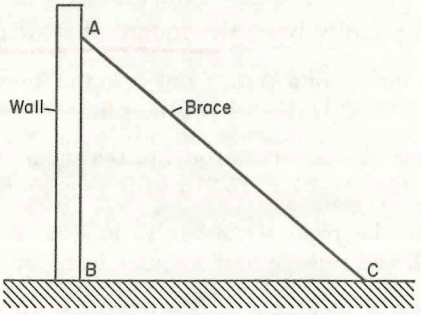


FIG. 4

SOLUTION. The solution of this example can be aided by the sketch in Fig. 4. The vertical line *AB* represents the wall and *AC* the brace. The height *AB* is 20 ft, and the distance *BC* is 15 ft. You are asked to find the length of *AC*. Since *AC* is the longest side of a right triangle, its square is equal to the sum of the squares of the shorter sides, or $15^2 + 20^2 = 225 + 400 = 625$. If the square of *AC* is 625, its length is $\sqrt{625} = 25$ ft. Ans.

EXAMPLE 3. A ladder 40 ft long stands against a wall with its lower end 8 ft

from the wall. How far, to the nearest tenth of a foot, is the top of the ladder from the ground?

SOLUTION. Sketch the arrangement and you will see that the ladder, the wall, and the level ground form a right triangle in which the lengths of one short side and the longest side are known. As we stated in Art. 67, the length of one short side of a right triangle can be found by squaring the longest side, subtracting the square of the other short side, and then extracting the square root of the difference. Thus, the height of the top of the ladder above the ground is $\sqrt{40^2 - 8^2} = \sqrt{1600 - 64} = \sqrt{1536}$ = 39.19 or 39.2 ft. Ans.

Finding the Side of a Square and the Diameter of a Circle

68. We have already seen that the area of a square is equal to the square of one of its sides. It is logical, therefore, that you would take the square root of the area to find the side of a square. Likewise, we learned that the area of a circle is 0.7854 times the square of its diameter. Again, reversing this procedure will give us the diameter of a circle if its area is known. In other words, we divide the area by 0.7854 and then take the square root of the quotient. This result will give us the required diameter. The following examples will show you how this work is done.

EXAMPLE 1. A square plot of land contains 1210 sq yd (square yards). What is the length of one side to the nearest tenth of a yard?

SOLUTION. The side of a square has a length equal to the square root of the area. Hence, the side of the plot is $\sqrt{1210} = 34.78$ or 34.8 yd. Ans.

EXAMPLE 2. A circle has an area of 2064 sq ft. What is its diameter?

SOLUTION. First, divide the area by 0.7854. Thus, 2064 ÷ 0.7854 = 2628. Then, the diameter is $\sqrt{2628}$, or 51.3 ft. Ans.

EXAMPLE 3. The area of a circular steel plate is 50 sq in. Find the diameter of the plate to one decimal place.

SOLUTION. Divide the area by 0.7854. Thus, 50 ÷ 0.7854 = 63.66 sq in. The required diameter is $\sqrt{63.66}$ = 7.97 or 8.0 in. Ans.

Notice that we retained two decimal places in our solution. We did this so that we could be sure of the first decimal place.

Practice Problems

1. What must be the length of a ladder to reach to the top of a wall 40 ft high, if the foot of the ladder is 9 ft from the wall?
2. The area of a square is 5,776 sq ft. What is the length of its side?
3. A rectangular room is 24 ft long and 18 ft wide. What is the diagonal distance from corner to corner?
4. The opening of a safety valve on a steam boiler has an area of 12 sq in. What is the diameter of the opening? Express the answer to two decimal places.

5. A circular plate of brass has an area of 100 sq in. What is its diameter? Express the answer to two decimal places.

Number Systems

Our Number System

69. Very early in Part 1 of *Practical Arithmetic* we spoke of "our number system." We learned how to count or how integers increase in value. Then we discussed the four basic operations and how they apply to integers. It was not very long until we discovered that our system of integers was not adequate to handle all the problems that occurred in even the simplest situations. This discovery led to the study of fractions and decimals. At this point our system of numbers seemed to be complete. However, even though we have learned much about numbers, there is much more to be learned about our number system.

Some of the Features of Our Number System

70. We have learned about the simple and relative values of the digits in a number and the importance of their place values in that number. In our system of writing numbers it takes 10 ones to make 1 ten, 10 tens to make 1 hundred, 10 hundreds to make 1 thousand, and so forth. We observed that moving the decimal point to the right in a number multiplies the number by 10, 100, or 1000 depending upon how many places it is moved. Later on we found that to divide a number by 10, 100, or 1000, the decimal point is moved to the left. From all this, you should begin to suspect that moving a digit in a number one place to the left multiplies its value by 10 and moving a digit in a number to the right divides its value by 10. These are some of the features of our number system that will be explained in the next few articles.

Why Ten Is the Base for Our Number System

71. Let's examine the number 3333. The 3 at the far right of this number is in the units place and has a value of 3, the next 3 to the left is in the tens place, so that it has a value of 3 tens. As we continue to the left the third 3 is in the hundreds place and its value is 3 hundreds. Finally, the fourth 3 is in the thousands place and has a value of 3 thousands. Therefore, we can write 3333 as

$$3 \times 1000 + 3 \times 100 + 3 \times 10 + 3 \times 1$$

Recalling that $10 \times 1 = 10$; that $100 = 10 \times 10$ and that $1000 = 10 \times 100$, we can observe that 3×10 (the tens digit) is ten times 3×1 (the units digit) and 3×100 is ten times 3×10. In other words, the value of each digit is ten times the value of the digit to its right. This means that we can write the preceding expression in this manner

$$\begin{aligned} 3333 &= 3 \times 10 \times 100 + 3 \times 10 \times 10 + 3 \times 10 \times 1 + 3 \times 1 \\ &= 3 \times 10 \times 10 \times 10 + 3 \times 10 \times 10 + 3 \times 10 + 3 \times 1 \end{aligned}$$

By using what we learned about exponents it is possible to write 3333 also as

$$3 \times 10^3 + 3 \times 10^2 + 3 \times 10^1 + 3 \times 10^0$$

Here the only parts of this expression you might wonder about are 10^1 and 10^0. However, after a little reflection on your part, you should be able to understand why $10^1 = 10$. Since 10^2 means that 10 is a factor twice or 10×10, 10^1 should mean that 10 is a factor only once or just 10 alone. The explanation of why $10^0 = 1$ is not quite as simple. Just take our word that it is true. Later on, if you study algebra, you will see why it is true. Remember these facts:

1. Any number raised to the first power is the number itself.
2. Any number raised to the zero power is 1.

When we write 3333 as $3 \times 10^3 + 3 \times 10^2 + 3 \times 10^1 + 3 \times 10^0$, it is clear that the number 10 plays a very important role. In fact, we say that 10 is the *base* of our number system because 10 is the *base number* for the exponents. Surely you have heard persons refer to our number system as the decimal system. The word "decimal" comes from the Latin word *decem*, which means ten.

How to Write a Number to Any Base

72. The method of writing a decimal system number as powers of 10 is most important, and you should be sure that you understand it. This same idea is the key to writing numbers with any base. For example, if you wanted to express the value of the number 31045 with the base 7, you would write this:

$$3 \times 7^4 + 1 \times 7^3 + 0 \times 7^2 + 4 \times 7^1 + 5 \times 7^0$$

Soon you will learn how to express the value of this number in the decimal system, but for the present concentrate on how it is written.

To indicate that 31045 is a number whose base is 7, it is written like this: 31045_7. Notice that the base is indicated by the small 7 written to the right of and slightly below the last digit. When no base number is shown, it will be assumed that the number is a base 10, or decimal, number.

The following summarizes what we have learned to this point:

	Base Number
$8647 = 8 \times 10^3 + 6 \times 10^2 + 4 \times 10^1 + 7 \times 10^0$	10
$6435_7 = 6 \times 7^3 + 4 \times 7^2 + 3 \times 7^1 + 5 \times 7^0$	7
$4321_5 = 4 \times 5^3 + 3 \times 5^2 + 2 \times 5^1 + 1 \times 5^0$	5
$2101_3 = 2 \times 3^3 + 1 \times 3^2 + 0 \times 3^1 + 1 \times 3^0$	3
$1010_2 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$	2

In reading and writing numbers with bases other than 10 we have to be careful not to read them by using our familiar decimal language. For example, the number 6435_7 should not be read as six thousand four hundred thirty-five. You should read 6435_7 like this: six four three five with base 7.

Check Your Learning

102. If we move the decimal point in a number one place to the right, the number is multiplied by _____.
(10)

103. If we move the decimal point in a number one place to the left, the number is divided by _____.
(10)

104. Moving a digit in a number one place to the _____ multiplies its value by 10.
(left)

105. Moving a digit in a number one place to the _____ divides its value by 10.
(right)

106. In the number 777, the 7 at the far right has a value of _____.
(7)

107. In the number 777, the middle 7 has a value of 7 _____.
(tens)

108. In the number 777, the 7 at the far left has a value of 7 _____.
(hundreds)

109. The number 777 may be written as $7 \times$ _____ $+ 7 \times$ _____ $+ 7 \times$ _____.
(100, 10, 1)

110. The value 7 hundreds written as a power of 10 is $7 \times$ _____.
(10^2)

111. The number 777 written as powers of 10 is $7 \times$ ____ $+ 7 \times$ ____ $+ 7 \times$ ____.
(10^2, 10^1, 10^0)

112. From the preceding answer, it is clear that the number _____ plays an important role in our number system.
(10)

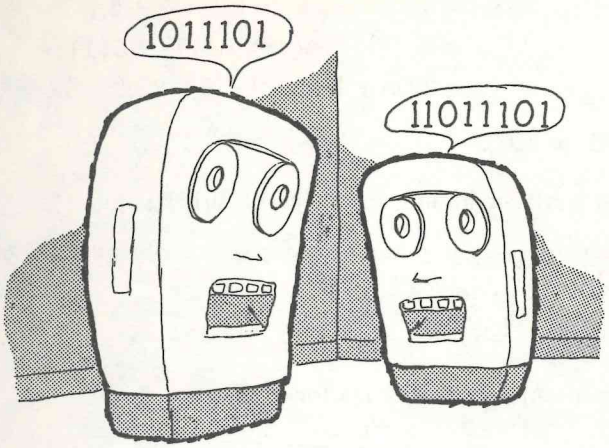
113. We say that 10 is the _____ of our number system.
(base)

114. The base 10 number system is sometimes called the _____ system.
(decimal)

115. To indicate that 6523 is a number whose base is 8, write it as _____.
(6523_8)

116. You should read 6523_8 as six five two three _____ _____ _____.
(with base 8)

117. Expressing the value of the number 6523_8, you would write $6 \times$ _____ $+ 5 \times$ _____ $+ 2 \times$ _____ $+ 3 \times$ _____.
(8^3, 8^2, 8^1, 8^0)



Practice Problems

1. Write each of the following numbers using powers of the given base.

a) 20,540 *b)* 7654_8 *c)* 678 *d)* 4343_5 *e)* 4762 *f)* $52{,}432_6$

The Binary Number System

73. The preceding explanations have done more than introduce us to new number systems. They have also made our decimal system more understandable, which is one of the major reasons for discussing them. There is, however, another very important reason for our discussion. Everyone in this present day of automation and computers should be aware of the role of "other" number systems in our lives. Computing machines, or computers, usually "understand," or work with, numbers with bases other than 10. The most common base used in machines is 2. However, some machines use 8, 16, or 64 as bases.

The reason why 2 is so common as a base is that many events or situations can be made to depend upon successive dual choices of yes or no; on or off; present or absent; and so on. These choices can be represented by 0 and 1, which are the digits in a binary system of numbers. The binary system is often called the language of computers. A binary number looks like this: 10101_2. It can be written in the expanded form, as we learned before. In other words

$$10101_2 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

Finding the Value of a Number with Any Base

74. We are sure that you have already figured out how to find the value of any number with any base. When we speak of finding the value of a number with any base it should be obvious that we want an equivalent base 10 number. For example, to find the value of 6435_7 we must find a base 10 number that is equal to it. Our work would look like this.

$$6435_7 = 6 \times 7^3 + 4 \times 7^2 + 3 \times 7^1 + 5 \times 7^0$$
$$= 6(343) + 4(49) + 3(7) + 5(1)$$
$$= 2058 + 196 + 21 + 5 = 2280$$

Therefore, $6435_7 = 2280$.

Here is the work for finding the value of 10101_2:

$$10101_2 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$
$$= 16 + 0 + 4 + 0 + 1$$
$$= 21$$

Here are some more conversions for you to try.

Practice Problems

1. Give the values of the following numbers.

a) 23421_5 *b)* 1001_2 *c)* 3214_6 *d)* 323_4 *e)* 6523_8

Changing a Base Ten Number to Another Base

75. It is quite a simple matter to change a number with any base to a base 10 number. After you have performed some of the expansions such as were given in the preceding articles, you will appreciate the ease of working with decimal numbers. The fact that any power of 10 is so simple to calculate is the reason why the decimal numbers are best for any calculation you will be required to make. However, just as it was necessary to know the base 10 value of a base 2 number, for example, so it is necessary to know how to write the base 2 value of a base 10 number. Before showing how this is done, we should mention something that you probably have noticed concerning the digits in a number. The digits in a base 10 number are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. For a base 2 number the digits are 0 and 1. These examples illustrate a general rule. *The largest digit in any number system is one unit less than the base number.* Thus we know that for a base 7 number the digits will be 0, 1, 2, 3, 4, 5, and 6.

The easiest method of converting a decimal number to a number with any other base is quite simple to use. We call it the *remainder method.* This is the procedure:

Divide the base 10 number by the *base* of the new system. The remainder for each of the divisions will be a digit for the new number, beginning with the units digit. An example will make the procedure clear.

EXAMPLE 1. Change 857 to a base 5 number.

SOLUTION. First divide 857 by 5. You get 171 and a remainder of 2. This first remainder, 2, will be the units digit of the base 5 number. Now continue to divide, and write each remainder in the next place. When you divide 171 by 5, you get 34 and a remainder of 1. This remainder is written to the left of the previous remainder, 2.

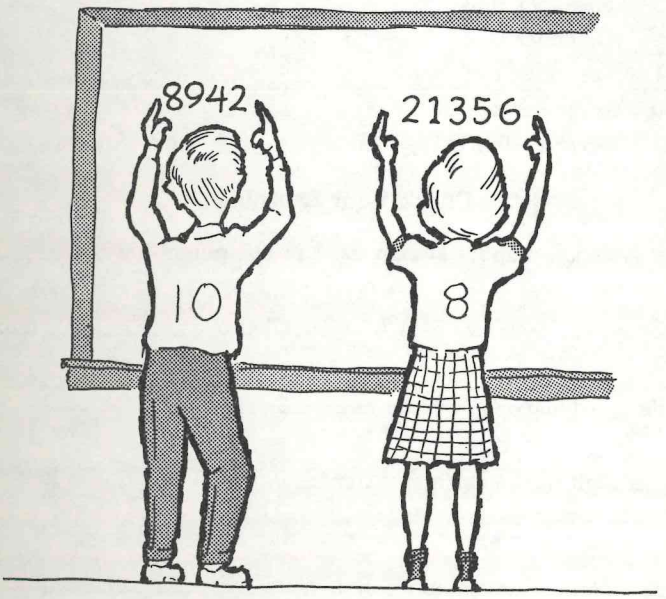
When you divide 34 by 5, you get 6 and a remainder of 4. Divide 6 by 5, and you get 1 and a remainder of 1. Finally, 1 divided by 5 is 0, with a remainder of 1. The base 5 number is 11412_5. This is how the work will look.

5/857
5/171 remainder 2 (first remainder, units place)
5/34 remainder 1
5/6 remainder 4
5/1 remainder 1
0 remainder 1

$857 = 11412_5$ Ans.

Check your work by changing 11412_5 to a base 10 number.

$$11412_5 = 1 \times 5^4 + 1 \times 5^3 + 4 \times 5^2 + 1 \times 5^1 + 2 \times 5^0$$
$$= 1 \times 625 + 1 \times 125 + 4 \times 25 + 1 \times 5 + 2 \times 1$$
$$= 625 + 125 + 100 + 5 + 2 = 857$$
$$11412_5 = 857$$



EXAMPLE 2. Change 8942 to a base 8 number.

SOLUTION.

8/8942
8/1117 remainder 6 (first remainder, units place)
8/139 remainder 5
8/17 remainder 3
8/2 remainder 1
0 remainder 2

$8942 = 21356_8$ Ans.

Check by changing 21356_8 to a base 10 number.

$$\begin{aligned} 21356_8 &= 2 \times 8^4 + 1 \times 8^3 + 3 \times 8^2 + 5 \times 8^1 + 6 \times 8^0 \\ &= 2 \times 4096 + 1 \times 512 + 3 \times 64 + 5 \times 8 + 6 \times 1 \\ &= 8192 + 512 + 192 + 40 + 6 = 8942 \\ 21356_8 &= 8942 \end{aligned}$$

EXAMPLE 3. Change 783 to a base 2 number.

SOLUTION.

2/783	
2/391	remainder 1 (first remainder, units place)
2/195	remainder 1
2/97	remainder 1
2/48	remainder 1
2/24	remainder 0
2/12	remainder 0
2/6	remainder 0
2/3	remainder 0
2/1	remainder 1
0	remainder 1

Don't forget these zeros; they are important!

$783 = 1100001111_2$ Ans.

Check Your Learning

118. The most common number system used in computers has _____ as its base.
(two)

119. A number system with two as a base is called a _____ number system.
(binary)

120. The digits in a binary system are _____ _____ _____.
(0 and 1)

121. The largest digit in the decimal system is 9, which is one unit less than the base number _____.
(10)

122. The largest digit in any number system is _____ _____ _____ than the base number.
(one unit less)

123. To convert 5679 to a number with base 5, divide 5679 repeatedly by _____.
(5)

124. To convert a decimal number to a number with any other base, divide the base 10 number by the _____ of the new system.
(base)

Practice Problems

1. Change 42,467 to a base 8 number.
2. Change 4310 to a base 5 number.
3. Change 1686 to a base seven number.
4. Change 2865 to a base six number.
5. Change 492 to a base three number.

Answers to Practice Problems

Article 10

1. *a)* 1024 *b)* 1692 *c)* 842 *d)* 8645 *e)* 148,361

2. *a)* 5135

Sub-totals
1522
1960
1653
5135

b) 53,082

Sub-totals
15891
26287
10904
53082

c) 31,050

Sub-totals
8725
15318
7007
31050

d) 30,889

1650
12840
16399
30889

e) 24,554

8657
1414
14483
24554

3. *a)* 40 (40) *b)* 967 (1000) *c)* 15,464 (15,000) *d)* 10,487 (10,700) *e)* 5923 (6160)
4. *a)* 575 *b)* 548 *c)* 1499 *d)* 965 *e)* 5709
5. *a)* 45,890 (50,000) *b)* 26,571 (20,000) *c)* 36,476 (40,000) *d)* 38,364 (40,000) *e)* 589,993 (600,000)
6. 362,310 people
7. 480,166 books (480,000)
8. 6615 miles (7,000) miles

Article 14

1. 1850 2. 2048 3. 277,900 4. 42777 5. 503,400 6. 62

Article 21

1. *a)* 39700 *b)* 778,400 *c)* 13,770 *d)* 30,000 *e)* 35,742 *f)* 22,517
2. *a)* 166,254 (140,000) *b)* 57,519 (56,000) *c)* 414,960 (450,000) *d)* 32,034 (32,000) *e)* 36,401 (36,000) *f)* 41,702 (42,000)
3. *a)* 28 (30) *b)* 343 (300) *c)* 6468 (6000) *d)* 2450.8 *e)* 136.2 *f)* 73.4
4. 2148 days
5. $30

Article 25

1. 2 2. 4 3. 4 4. 7 5. 8

Article 26

1. 1 2. 7 3. 9 4. 7 5. 5

Article 27

1. *a)* 19160 *b)* 2758 *c)* 1629 *d)* 20,094 *e)* 23,525
2. *a)* 52,704 *b)* 4,416,300 *c)* 195,662 *d)* 621,270 *e)* 175,273
3. *a)* 133 remainder 53 *b)* 81 remainder 407 *c)* 3466 remainder 94
   d) 2804 remainder 168 *e)* 2941 remainder 29

Article 29

1. 9 2. 2 3. 17 4. 11 5. 15

Article 31

1. 13 2. 62 3. 15 4. 6 5. 41

Article 32

1. 16 2. 29 3. 0 4. 3 5. 15

Article 45

1. *a)* 65% *b)* 37.5% *c)* .75% *d)* 61%
2. *a)* .30 *b)* .004 *c)* .025 *d)* .1357
3. *a)* .069 *b)* .412 *c)* .231 *d)* .640
4. *a)* 3¢ *b)* \$1.18
5. 180

Article 48

1. *a)* 9 *b)* 2.25 *c)* 0.64 *d)* 0.0004
2. *a)* 1024 *b)* $\frac{16}{25}$ *c)* $\frac{64}{9}$ *d)* 1600
3. *a)* 343 *b)* 1.728 *c)* 0.216 *d)* 0.000064
4. *a)* 13,824 *b)* $\frac{8}{27}$ *c)* $\frac{125}{64}$ *d)* 27,000

Article 50

1. *a)* 3^2 *b)* 15^2 *c)* 0.8^2 *d)* 0.02^2
2. *a)* 7^3 *b)* 1.2^3 *c)* 0.6^3 *d)* 0.04^3
3. *a)* 5×5
   b) 16.4×16.4
   c) 0.7×0.7
   d) $11 \times 11 \times 11$
   e) $6.2 \times 6.2 \times 6.2$
   f) $0.03 \times 0.03 \times 0.03$
   g) $6 \times 6 \times 6 \times 6$
   h) $8.9 \times 8.9 \times 8.9 \times 8.9 \times 8.9$
4. *a)* Five squared
   b) Sixteen point four squared
   c) Zero point seven squared
   d) Eleven cubed
   e) Six point two cubed
   f) Zero point zero three cubed
   g) Six to the fourth power
   h) Eight point nine to the fifth power

Article 53

1. *a)* $\frac{1}{343}$ *b)* $\frac{16}{81}$ *c)* $\frac{1024}{3125}$ *d)* $\frac{343}{1331}$
2. *a)* 0.49 *b)* 0.0121 *c)* 0.0441 *d)* 0.000324

Article 58

1. 19.18 2. 28.05 3. 23.58 4. 25.20 5. 29.56

Article 62

1. 37 2. 14 3. 57 4. 208 5. 369 6. 174 7. 1075 8. 4444

Article 64

1. 43.2 2. 0.108 3. 14.077 4. 14.32 5. 1.541 6. 2.138 7. 10.392 8. 0.077

Article 65

1. 1.4 2. 1.803 3. 3.010 4. 0.913 5. 0.655 6. 0.395

Article 68

1. 41 ft 2. 76 ft 3. 30 ft 4. 3.91 in. 5. 11.28

Article 72

1. *a)* $2 \times 10^4 + 0 \times 10^3 + 5 \times 10^2 + 4 \times 10^1 + 0 \times 10^0$
   b) $7 \times 8^3 + 6 \times 8^2 + 5 \times 8^1 + 4 \times 8^0$
   c) $6 \times 10^2 + 7 \times 10^1 + 8 \times 10^0$
   d) $4 \times 5^3 + 3 \times 5^2 + 4 \times 5^1 + 3 \times 5^0$
   e) $4 \times 10^3 + 7 \times 10^2 + 6 \times 10^1 + 2 \times 10^0$
   f) $5 \times 6^4 + 2 \times 6^3 + 4 \times 6^2 + 3 \times 6^1 + 2 \times 6^0$

Article 74

1. *a)* 1736 *b)* 9 *c)* 730 *d)* 59 *e)* 3411

Article 75

1. 122743_8 2. 114220_5 3. 4626_7 4. 21133_6 5. 200020_3

TABLE 3

SQUARE ROOTS OF NUMBERS

Number	Square Root	Number	Square Root	Number	Square Root	Number	Square Root	Number	Square Root
1	1.000	41	6.403	81	9.000	121	11.00	161	12.69
2	1.414	42	6.481	82	9.055	122	11.05	162	12.73
3	1.732	43	6.557	83	9.110	123	11.09	163	12.77
4	2.000	44	6.633	84	9.165	124	11.14	164	12.81
5	2.236	45	6.708	85	9.220	125	11.18	165	12.85
6	2.449	46	6.782	86	9.274	126	11.22	166	12.88
7	2.646	47	6.856	87	9.327	127	11.27	167	12.92
8	2.828	48	6.928	88	9.381	128	11.31	168	12.96
9	3.000	49	7.000	89	9.434	129	11.36	169	13.00
10	3.162	50	7.071	90	9.487	130	11.40	170	13.04
11	3.317	51	7.141	91	9.539	131	11.45	171	13.08
12	3.464	52	7.211	92	9.592	132	11.49	172	13.11
13	3.606	53	7.280	93	9.644	133	11.53	173	13.15
14	3.742	54	7.349	94	9.695	134	11.58	174	13.19
15	3.873	55	7.416	95	9.747	135	11.62	175	13.23
16	4.000	56	7.483	96	9.798	136	11.66	176	13.27
17	4.123	57	7.550	97	9.849	137	11.70	177	13.30
18	4.243	58	7.616	98	9.900	138	11.75	178	13.34
19	4.359	59	7.681	99	9.950	139	11.79	179	13.38
20	4.472	60	7.746	100	10.00	140	11.83	180	13.42
21	4.583	61	7.810	101	10.05	141	11.87	181	13.45
22	4.690	62	7.874	102	10.10	142	11.92	182	13.49
23	4.796	63	7.937	103	10.15	143	11.96	183	13.53
24	4.899	64	8.000	104	10.20	144	12.00	184	13.56
25	5.000	65	8.062	105	10.25	145	12.04	185	13.60
26	5.099	66	8.124	106	10.30	146	12.08	186	13.64
27	5.196	67	8.185	107	10.34	147	12.12	187	13.67
28	5.292	68	8.246	108	10.39	148	12.17	188	13.71
29	5.385	69	8.307	109	10.44	149	12.21	189	13.75
30	5.477	70	8.367	110	10.49	150	12.25	190	13.78
31	5.568	71	8.426	111	10.54	151	12.29	191	13.82
32	5.657	72	8.485	112	10.58	152	12.33	192	13.86
33	5.745	73	8.544	113	10.63	153	12.37	193	13.89
34	5.831	74	8.602	114	10.68	154	12.41	194	13.93
35	5.916	75	8.660	115	10.72	155	12.45	195	13.96
36	6.000	76	8.718	116	10.77	156	12.49	196	14.00
37	6.083	77	8.775	117	10.82	157	12.53	197	14.04
38	6.164	78	8.832	118	10.86	158	12.57	198	14.07
39	6.245	79	8.888	119	10.91	159	12.61	199	14.11
40	6.325	80	8.944	120	10.95	160	12.65	200	14.14

TABLE 3 – *Continued*

Number	Square Root	Number	Square Root	Number	Square Root	Number	Square Root	Number	Square Root
201	14.18	241	15.52	281	16.76	321	17.92	361	19.00
202	14.21	242	15.56	282	16.79	322	17.94	362	19.03
203	14.25	243	15.59	283	16.82	323	17.97	363	19.05
204	14.28	244	15.62	284	16.85	324	18.00	364	19.08
205	14.32	245	15.65	285	16.88	325	18.03	365	19.10
206	14.35	246	15.68	286	16.91	326	18.06	366	19.13
207	14.39	247	15.72	287	16.94	327	18.08	367	19.16
208	14.42	248	15.75	288	16.97	328	18.11	368	19.18
209	14.46	249	15.78	289	17.00	329	18.14	369	19.21
210	14.49	250	15.81	290	17.03	330	18.17	370	19.24
211	14.53	251	15.84	291	17.06	331	18.19	371	19.26
212	14.56	252	15.87	292	17.09	332	18.22	372	19.29
213	14.59	253	15.91	293	17.12	333	18.25	373	19.31
214	14.63	254	15.94	294	17.15	334	18.28	374	19.34
215	14.66	255	15.97	295	17.18	335	18.30	375	19.36
216	14.70	256	16.00	296	17.20	336	18.33	376	19.39
217	14.73	257	16.03	297	17.23	337	18.36	377	19.42
218	14.76	258	16.06	298	17.26	338	18.38	378	19.44
219	14.80	259	16.09	299	17.29	339	18.41	379	19.47
220	14.83	260	16.12	300	17.32	340	18.44	380	19.49
221	14.87	261	16.16	301	17.35	341	18.47	381	19.52
222	14.90	262	16.19	302	17.38	342	18.49	382	19.54
223	14.93	263	16.22	303	17.41	343	18.52	383	19.57
224	14.97	264	16.25	304	17.44	344	18.55	384	19.60
225	15.00	265	16.28	305	17.46	345	18.57	385	19.62
226	15.03	266	16.31	306	17.49	346	18.60	386	19.65
227	15.07	267	16.34	307	17.52	347	18.63	387	19.67
228	15.10	268	16.37	308	17.55	348	18.65	388	19.70
229	15.13	269	16.40	309	17.58	349	18.68	389	19.72
230	15.17	270	16.43	310	17.61	350	18.71	390	19.75
231	15.20	271	16.46	311	17.64	351	18.73	391	19.77
232	15.23	272	16.49	312	17.66	352	18.76	392	19.80
233	15.26	273	16.52	313	17.69	353	18.79	393	19.82
234	15.30	274	16.55	314	17.72	354	18.81	394	19.85
235	15.33	275	16.58	315	17.75	355	18.84	395	19.87
236	15.36	276	16.61	316	17.78	356	18.87	396	19.90
237	15.39	277	16.64	317	17.80	357	18.89	397	19.92
238	15.43	278	16.67	318	17.83	358	18.92	398	19.95
239	15.46	279	16.70	319	17.86	359	18.95	399	19.97
240	15.49	280	16.73	320	17.89	360	18.97	400	20.00

TABLE 3 – *Continued*

Number	Square Root	Number	Square Root	Number	Square Root	Number	Square Root	Number	Square Root
401	20.02	441	21.00	481	21.93	521	22.83	561	23.69
402	20.05	442	21.02	482	21.95	522	22.85	562	23.71
403	20.07	443	21.05	483	21.98	523	22.87	563	23.73
404	20.10	444	21.07	484	22.00	524	22.89	564	23.75
405	20.12	445	21.10	485	22.02	525	22.91	565	23.77
406	20.15	446	21.12	486	22.05	526	22.93	566	23.79
407	20.17	447	21.14	487	22.07	527	22.96	567	23.81
408	20.20	448	21.17	488	22.09	528	22.98	568	23.83
409	20.22	449	21.19	489	22.11	529	23.00	569	23.85
410	20.25	450	21.21	490	22.14	530	23.02	570	23.87
411	20.27	451	21.24	491	22.16	531	23.04	571	23.90
412	20.30	452	21.26	492	22.18	532	23.07	572	23.92
413	20.32	453	21.28	493	22.20	533	23.09	573	23.94
414	20.35	454	21.31	494	22.23	534	23.11	574	23.96
415	20.37	455	21.33	495	22.25	535	23.13	575	23.98
416	20.40	456	21.35	496	22.27	536	23.15	576	24.00
417	20.42	457	21.38	497	22.29	537	23.17	577	24.02
418	20.45	458	21.40	498	22.32	538	23.19	578	24.04
419	20.47	459	21.42	499	22.34	539	23.22	579	24.06
420	20.49	460	21.45	500	22.36	540	23.24	580	24.08
421	20.52	461	21.47	501	22.38	541	23.26	581	24.10
422	20.54	462	21.49	502	22.41	542	23.28	582	24.12
423	20.57	463	21.52	503	22.43	543	23.30	583	24.15
424	20.59	464	21.54	504	22.45	544	23.32	584	24.17
425	20.62	465	21.56	505	22.47	545	23.35	585	24.19
426	20.64	466	21.59	506	22.49	546	23.37	586	24.21
427	20.66	467	21.61	507	22.52	547	23.39	587	24.23
428	20.69	468	21.63	508	22.54	548	23.41	588	24.25
429	20.71	469	21.66	509	22.56	549	23.43	589	24.27
430	20.74	470	21.68	510	22.58	550	23.45	590	24.29
431	20.76	471	21.70	511	22.61	551	23.47	591	24.31
432	20.78	472	21.73	512	22.63	552	23.49	592	24.33
433	20.81	473	21.75	513	22.65	553	23.52	593	24.35
434	20.83	474	21.77	514	22.67	554	23.54	594	24.37
435	20.86	475	21.79	515	22.69	555	23.56	595	24.39
436	20.88	476	21.82	516	22.72	556	23.58	596	24.41
437	20.90	477	21.84	517	22.74	557	23.60	597	24.43
438	20.93	478	21.86	518	22.76	558	23.62	598	24.45
439	20.95	479	21.89	519	22.78	559	23.64	599	24.47
440	20.98	480	21.91	520	22.80	560	23.66	600	24.49

TABLE 3 — *Continued*

Number	Square Root	Number	Square Root	Number	Square Root	Number	Square Root	Number	Square Root
601	24.52	641	25.32	681	26.10	721	26.85	761	27.59
602	24.54	642	25.34	682	26.12	722	26.87	762	27.60
603	24.56	643	25.36	683	26.13	723	26.89	763	27.62
604	24.58	644	25.38	684	26.15	724	26.91	764	27.64
605	24.60	645	25.40	685	26.17	725	26.93	765	27.66
606	24.62	646	25.42	686	26.19	726	26.94	766	27.68
607	24.64	647	25.44	687	26.21	727	26.96	767	27.69
608	24.66	648	25.46	688	26.23	728	26.98	768	27.71
609	24.68	649	25.48	689	26.25	729	27.00	769	27.73
610	24.70	650	25.50	690	26.27	730	27.02	770	27.75
611	24.72	651	25.51	691	26.29	731	27.04	771	27.77
612	24.74	652	25.53	692	26.31	732	27.06	772	27.78
613	24.76	653	25.55	693	26.32	733	27.07	773	27.80
614	24.78	654	25.57	694	26.34	734	27.09	774	27.82
615	24.80	655	25.59	695	26.36	735	27.11	775	27.84
616	24.82	656	25.61	696	26.38	736	27.13	776	27.86
617	24.84	657	25.63	697	26.40	737	27.15	777	27.87
618	24.86	658	25.65	698	26.42	738	27.17	778	27.89
619	24.88	659	25.67	699	26.44	739	27.18	779	27.91
620	24.90	660	25.69	700	26.46	740	27.20	780	27.93
621	24.92	661	25.71	701	26.48	741	27.22	781	27.95
622	24.94	662	25.73	702	26.50	742	27.24	782	27.96
623	24.96	663	25.75	703	26.51	743	27.26	783	27.98
624	24.98	664	25.77	704	26.53	744	27.28	784	28.00
625	25.00	665	25.79	705	26.55	745	27.29	785	28.02
626	25.02	666	25.81	706	26.57	746	27.31	786	28.04
627	25.04	667	25.83	707	26.59	747	27.33	787	28.05
628	25.06	668	25.85	708	26.61	748	27.35	788	28.07
629	25.08	669	25.87	709	26.63	749	27.37	789	28.09
630	25.10	670	25.88	710	26.65	750	27.39	790	28.11
631	25.12	671	25.90	711	26.66	751	27.40	791	28.12
632	25.14	672	25.92	712	26.68	752	27.42	792	28.14
633	25.16	673	25.94	713	26.70	753	27.44	793	28.16
634	25.18	674	25.96	714	26.72	754	27.46	794	28.18
635	25.20	675	25.98	715	26.74	755	27.48	795	28.20
636	25.22	676	26.00	716	26.76	756	27.50	796	28.21
637	25.24	677	26.02	717	26.78	757	27.51	797	28.23
638	25.26	678	26.04	718	26.80	758	27.53	798	28.25
639	25.28	679	26.06	719	26.81	759	27.55	799	28.27
640	25.30	680	26.08	720	26.83	760	27.57	800	28.28

TABLE 3 – *Continued*

Number	Square Root	Number	Square Root	Number	Square Root	Number	Square Root	Number	Square Root
801	28.30	841	29.00	881	29.68	921	30.35	961	31.00
802	28.32	842	29.02	882	29.70	922	30.36	962	31.02
803	28.34	843	29.03	883	29.72	923	30.38	963	31.03
804	28.35	844	29.05	884	29.73	924	30.40	964	31.05
805	28.37	845	29.07	885	29.75	925	30.41	965	31.06
806	28.39	846	29.09	886	29.77	926	30.43	966	31.08
807	28.41	847	29.10	887	29.78	927	30.45	967	31.10
808	28.43	848	29.12	888	29.80	928	30.46	968	31.11
809	28.44	849	29.14	889	29.82	929	30.48	969	31.13
810	28.46	850	29.15	890	29.83	930	30.50	970	31.14
811	28.48	851	29.17	891	29.85	931	30.51	971	31.16
812	28.50	852	29.19	892	29.87	932	30.53	972	31.18
813	28.51	853	29.21	893	29.88	933	30.55	973	31.19
814	28.53	854	29.22	894	29.90	934	30.56	974	31.21
815	28.55	855	29.24	895	29.92	935	30.58	975	31.22
816	28.57	856	29.26	896	29.93	936	30.59	976	31.24
817	28.58	857	29.27	897	29.95	937	30.61	977	31.26
818	28.60	858	29.29	898	29.97	938	30.63	978	31.27
819	28.62	859	29.31	899	29.98	939	30.64	979	31.29
820	28.64	860	29.33	900	30.00	940	30.66	980	31.30
821	28.65	861	29.34	901	30.02	941	30.68	981	31.32
822	28.67	862	29.36	902	30.03	942	30.69	982	31.34
823	28.69	863	29.38	903	30.05	943	30.71	983	31.35
824	28.71	864	29.39	904	30.07	944	30.72	984	31.37
825	28.72	865	29.41	905	30.08	945	30.74	985	31.38
826	28.74	866	29.43	906	30.10	946	30.76	986	31.40
827	28.76	867	29.44	907	30.12	947	30.77	987	31.42
828	28.77	868	29.46	908	30.13	948	30.79	988	31.43
829	28.79	869	29.48	909	30.15	949	30.81	989	31.45
830	28.81	870	29.50	910	30.17	950	30.82	990	31.46
831	28.83	871	29.51	911	30.18	951	30.84	991	31.48
832	28.84	872	29.53	912	30.20	952	30.85	992	31.50
833	28.86	873	29.55	913	30.22	953	30.87	993	31.51
834	28.88	874	29.56	914	30.23	954	30.89	994	31.53
835	28.90	875	29.58	915	30.25	955	30.90	995	31.54
836	28.91	876	29.60	916	30.27	956	30.92	996	31.56
837	28.93	877	29.61	917	30.28	957	30.94	997	31.58
838	28.95	878	29.63	918	30.30	958	30.95	998	31.59
839	28.97	879	29.65	919	30.32	959	30.97	999	31.61
840	28.98	880	29.66	920	30.33	960	30.98	1000	31.62

$$\frac{2.5}{100} \times \frac{32}{1} \quad \frac{12}{5} = 2.$$

$5/12/2.4$

$$\frac{112}{2} \times \frac{10}{1}$$

$2/112/56$

$10 - 6$

$= 4$

Practical Arithmetic

Serial 6750D-1 PART 4 Edition 1

Examination Questions

Notice to Students.—Study this instruction text thoroughly before you try to answer the following questions. Read each question carefully and be sure you understand it. Then write the best answer you can. If your answer requires a mathematical solution, show enough of your work to indicate the method you used. Unless you do this we will have no way of knowing how you obtained your anwser and your work will not be considered complete. After you have answered all the questions, double-check them to make sure that this is the best work you can do. Don't omit any questions.

Mail your work to us as soon as it is finished. DON'T HOLD IT until another examination is ready.

Each of the following questions is worth 10 points.

1. Estimate the answers to the following problems. Give only the approximate value for each number and the approximate answer you would obtain if you had performed the operations mentally.
   a) Find the sum of 17,842, 6471, 924, and 5993.
   b) What is the product of 25,487 and 33?
   c) Find the quotient when 14,978 is divided by 251.

2. *a*) The statement $2 + 3 = 3 + 2$ illustrates the _____ property of addition.
   b) A statement such as $(2 + 3) + 4 = 2 + (3 + 4)$ points out that addition possesses the _____ property.
   c) The expression $2(5 - 3)$ becomes $2 \times 5 - 2 \times 3$ by the _____ property of multiplication.
   d) When any number is multiplied by 1 the number remains unchanged. The special name for the number 1 because of this unique property is the _____ for multiplication.
   e) The equation $(7 \times 5) \times 4 = (7 \times 5) \times 4$ shows that multiplication possesses the _____ property.

3. *a*) What is $7\frac{1}{2}\%$ of 32? *b*) What is 112% of 50?

COMPLETE WORK IS REQUIRED ON QUESTIONS 4 TO 10.

4. Find the square root of each of the following. If the number is not a perfect square, express the root to two decimal places.
   a) 24.375 *b*) 0.004494

5. Find the square root of the fraction $\frac{7}{32}$ by first making the denominator a perfect square. Express the root to two decimal places.

6. A 20-foot ladder is leaning against a house and its base is 5 feet from the house. How far up the house will it reach? State your answer to the nearest foot.

7. The area of a circle is 75 sq ft. What is the diameter to the nearest $\frac{1}{10}$ of a foot?

8. Find the product of 8736 and 542. Check your work by casting out 9's and 11's. Note: In this question your work on the checking must be shown.

9. Remove grouping symbols and simplify the following:
   a) $4[7 + 2(4 - 2)] - 10 \times 2 + 6 \div 3$
   b) $(3 + 5)^2 + 12 - (4 - 2)^2 + 2[5 + 10 \div 2 - 3]$

10. *a*) Find the value of 11011_2.
    b) Change 37 to a base 2 number.